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Relating Green-Schwarz and extended pure spinor formalisms by similarity transformation

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Abstract: In order to gain deeper understanding of pure-spinor-based formalisms of superstring, an explicit similarity transformation is constructed which provides operator mapping between the light-cone Green-Schwarz (LCGS) formalism and the extended pure spinor (EPS) formalism, a recently proposed generalization of the Berkovits' formalism in an enlarged space. By applying a systematic procedure developed in our previous work, we first construct an analogous mapping in the bosonic string relating the BRST and the light-cone formulations. This provides sufficient insights and allows us to construct the desired mapping in the more intricate case of superstring as well. The success of the construction owes much to the enlarged field space where pure spinor constraints are removed and to the existence of the "B-ghost" in the EPS formalism.

Keywords: Superstrings and Heterotic Strings, Space-Time Symmetries, BRST Symmetry.

Contents

1. Introduction

Quantization of superstring in which both the Lorentz symmetry and the supersymmetry are manifest is a long standing problem of prime importance. Not long ago, a promising new approach based on the concept of pure spinor (PS) [1]–[3] was put forward by Berkovits [4], after many partially successful attempts of various kinds [5]–[11]. In this formulation the physical states of superstring are obtained as the cohomology of a BRST-like operator $Q_B = \int [dz] \lambda^\alpha d_\alpha$, where d_α is the spinor covariant derivative and λ^α is a bosonic chiral pure spinor satisfying the non-linear constraints $\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0$, which render Q_B nilpotent. All the independent worldsheet fields in this formalism are taken to be free and form a conformal field theory with vanishing center. This feature allows one to construct Q_B-invariant vertex operators explicitly and certain covariant rules are proposed to compute the scattering amplitudes, which reproduce known results [4],[12]–[14]. We refer the reader to [15]–[31] for further developments and [32] for a review up to a certain point.

Although highly compelling evidence has already been accumulated in support of the basic idea of this formulation, there are many points yet to be clarified and improved. Among them the most fundamental are the questions of the underlying action, its symmetries and how it should be quantized covariantly. In this regard it is ironic that the very concept of pure spinor with its characteristic quadratic constraints appears to cause some problems. Since only part of its components are independent free fields, Berkovits' formulation, in strict sense, is only U(5)-covariant. More importantly, the constrained Hilbert space is not large enough so that proper hermitean inner product is hard to define [29], the "B-ghost" needed to generate the energy-momentum tensor $T(z)$ via $\{Q_B, B(z)\} = T(z)$ is difficult to construct, etc. After all, such non-linear constraints would have to be avoided in quantization of the underlying action.

This motivated several groups to try to remove the PS constraints by introducing some finite number of compensating ghosts. One approach, described in a series of papers in [27], is to construct a new nilpotent BRST-like charge, enforcing covariance at every step, by several methods including one based on the gauged WZW structure. A price for the manifest covariance is that it appears non-trivial to assure the correct cohomology at the present stage.

An alternative formulation without PS constraints, which is in a sense complimentary to the above, was proposed in our previous work [29]. In this approach, to be briefly reviewed in section 2, equivalence with the Berkovits' cohomology is manifest by construction whereas the Lorentz symmetry is broken to U(5) in the newly added ghost sector. In this sense it is a minimal extension of the Berkovits' theory. The enlargement of the Hilbert space, however, has a definite advantage. An appropriate "B-ghost" field was obtained in a simple form using the added ghosts [29], and in a subsequent work [30], we have been able to construct a quantum similarity transformation which connects the BRST operator for the conventional RNS formalism to the one for the EPS without need of singular operations encountered in a previous attempt in the original PS formalism [13].

Although this explicit demonstration of the equivalence to the RNS via a similarity transformation added further credibility to the EPS formalism, the transformation was rather complicated due to the necessity of conversion between the spacetime vector ψ^μ in RNS and the spacetime spinor θ^α in EPS, which involved mixing of the RNS ghosts as well in an essential way. Also, since one has to use the formulation of RNS in the so-called "large" Hilbert space [33], the degeneracy due to "pictures" had to be carefully dealt with [13].

This suggests that it would be of great interest to try to connect the EPS formalism to the light-cone quantized Green-Schwarz (LCGS) formalism, where none of the above complications exists and the physical content of the theory is transparent. In fact at an early stage of the development of the PS formalism, such an attempt was made in [15]. In this work, SO(8) parametrization of the PS conditions is used, which turned out to be infinitely reducible. As a consequence, indefinite number of ghosts for ghosts had to be introduced and the demonstration of the equivalence with the light-cone formulation was quite cumbersome even conceptually.

In the present work, we shall make use of the systematic method developed in [30] to relate the LCGS formalism to the EPS formalism by constructing an appropriate similarity transformation. There are several advantages for making the connection by a similarity transformation. The function of such a transformation is to reorganize the Hilbert space into the direct product of physical and unphysical sectors $|\mathrm{phys}\rangle \otimes |\mathrm{unphys}\rangle$, thereby effectively achieving the gauge fixing without discarding any degrees of freedom (DOF). In other words, it emphasizes the aspect of decoupling of the unphysical DOF rather than their elimination. Moreover, in this process all the operatorial relations, in particular the (anti-)commutation relations and the symmetry structure, are preserved. We expect that the understanding gained by such a construction should prove quite useful for future attempts to uncover the underlying action and its symmetries.

Let us describe more explicitly the outline of our work. We begin with a brief review of the PS and the EPS formalisms in section 2. Then in section 3 we consider, as a warm up, the bosonic string and construct a similarity transformation which provides a mapping between the BRST and the light-cone formalisms. Besides proving the equivalence of two formulations in a manifest manner, this transformation explicitly converts the transverse oscillators α_n^i of the light-cone formalism into the DDF spectrum generating oscillators [34] A_n^i of the BRST formalism. Making use of the understanding gained by this exercise, we attack in section 4 our main problem of relating the LCGS and the EPS formalisms for superstring by a similarity mapping. This will be divided into three stages. In section 4.1, starting from the BRST version of the light-cone formalism we make two simple similarity transformations to bring the "light-cone BRST operator" $\bar{Q}$ into a form called $\tilde{Q}$, in which all the fields needed in EPS formalism are visible. Then in section 4.2, we develop further similarity transformations which simplify the BRST operator $\hat{Q}$ for the EPS formalism into $\tilde{Q}$. Finally in section 4.3 we show that in a suitably defined Hilbert space the cohomology of $\bar{Q}$ indeed gives the light-cone spectrum. We summarize our results and discuss future problems in section 5. Two appendices are provided to give some details not covered in the main text: In appendix A our conventions and some useful formulas are collected. Appendix B gives further details of the construction of the similarity transformations in the bosonic and the superstring cases.

2. A brief review of PS and EPS formalisms

To make this article reasonably self-contained and to explain our notations, let us begin with a brief review of the essential features of the PS and the EPS formalisms.

2.1 PS formalism

The central idea of the pure spinor formalism [4] is that the physical states of superstring can be described as the elements of the cohomology of a BRST-like operator Q_B given by[1]

$$Q_B = \int [dz] \lambda^\alpha(z) d_\alpha(z) \,, \tag{2.1}$$

where λ^α is a 16-component bosonic chiral spinor satisfying the pure spinor constraints

$$\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0 \,, \tag{2.2}$$

and d_α is the spinor covariant derivative given by

$$d_\alpha = p_\alpha + i\partial x_\mu (\gamma^\mu \theta)_\alpha + \frac{1}{2}(\gamma^\mu\theta)_\alpha(\theta\gamma_\mu\partial\theta) \,. \tag{2.3}$$

x^μ and θ^α are, respectively, the basic bosonic and ferminonic worldsheet fields describing a superstring, which transform under the spacetime supersymmetry with global spinor parameter ϵ^α as $\delta\theta^\alpha = \epsilon^\alpha$, $\delta x^\mu = i\epsilon\gamma^\mu\theta$. x^μ is self-conjugate and satisfies $x^\mu(z)x^\nu(w) = -\eta^{\mu\nu}\ln(z-w)$, while p_α serves as the conjugate to θ^α in the manner $\theta^\alpha(z)p_\beta(w) = \delta^\alpha_\beta(z-w)^{-1}$. θ^α and p_α carry conformal weights 0 and 1 respectively. With such free field operator product expansions (OPE's), d_α satisfies the following OPE with itself,

$$d_\alpha(z) d_\beta(w) = \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w} \,, \tag{2.4}$$

where Π_μ is the basic superinvariant combination

$$\Pi_\mu = \partial x_\mu - i\theta\gamma_\mu\partial\theta \,. \tag{2.5}$$

Then, due to the pure spinor constraints (2.2), Q_B is easily found to be nilpotent and the constrained cohomology of Q_B can be defined. The basic superinvariants d_α, Π^μ and $\partial\theta^\alpha$ form the closed algebra

$$\begin{aligned} d_\alpha(z) d_\beta(w) &= \frac{2i\gamma^\mu_{\alpha\beta}\Pi_\mu(w)}{z-w} \,, & (2.6) \\ d_\alpha(z) \Pi^\mu(w) &= \frac{-2i(\gamma^\mu\partial\theta)_\alpha(w)}{z-w} \,, & (2.7) \\ \Pi^\mu(z)\Pi^\nu(w) &= \frac{-\eta^{\mu\nu}}{(z-w)^2} \,, & (2.8) \\ d_\alpha(z)\partial\theta^\beta(w) &= \frac{\delta^\beta_\alpha}{(z-w)^2} \,, & (2.9) \end{aligned}$$

which has central charges and hence is essentially of *second class.*

Although eventually the rules for computing the scattering amplitudes are formulated in a Lorentz covariant manner, proper quantization of the pure spinor λ can only be performed by solving the PS constraints (2.2), which inevitably breaks covariance in intermediate steps. One convenient scheme is the so-called U(5) formalism,[2] in which a chiral

[1] For simplicity we will use the notation $[dz] \equiv dz/(2\pi i)$ throughout.

[2] Although our treatment applies equally well to SO(9, 1) and SO(10) groups, we shall use the terminology appropriate for SO(10), which contains U(5) as a subgroup. Some further explanation of our conventions for U(5) parametrization is given in appendix A.

and an anti-chiral spinors λ^α and χ_α, respectively, are decomposed in the following way

$$\lambda^\alpha = (\lambda_+, \lambda_{PQ}, \lambda_{\tilde{P}}) \sim (1, 10, \overline{5})\,, \tag{2.10}$$

$$\chi_\alpha = (\chi_-, \chi_{\tilde{P}\tilde{Q}}, \chi_P) \sim (1, \overline{10}, 5)\,, \qquad (P, Q, \tilde{P}, \tilde{Q} = 1 \sim 5)\,, \tag{2.11}$$

where we have indicated how they transform under U(5), with a tilde on the $\bar{5}$ indices. On the other hand, an arbitrary Lorentz vector u^μ is split into $5 + \overline{5}$ of U(5) as

$$u^\mu = 2(e_P^{+\mu} u_{\tilde{P}}^- + e_{\tilde{P}}^{-\mu} u_P^+)\,, \tag{2.12}$$

where the projectors $e_P^{\pm\mu}$, defined by

$$e_P^{\pm\mu} \equiv \frac{1}{2}(\delta_{\mu,2P-1} \pm i\delta_{\mu,2P})\,, \tag{2.13}$$

enjoy the properties

$$e_P^{\pm\mu} e_Q^{\pm\mu} = 0\,, \qquad e_P^{\pm\mu} e_Q^{\mp\mu} = \frac{1}{2}\delta_{PQ}\,, \tag{2.14}$$

$$e_P^{+\mu} e_{\tilde{P}}^{-\nu} + e_{\tilde{P}}^{-\mu} e_P^{+\nu} = \frac{1}{2}\delta^{\mu\nu}\,. \tag{2.15}$$

Thus, one can invert the relation (2.12) in the form

$$u_P^+ = e_P^{+\mu} u_\mu\,, \qquad u_{\tilde{P}}^- = e_{\tilde{P}}^{-\mu} u_\mu\,. \tag{2.16}$$

In this scheme the pure spinor constraints reduce to 5 independent conditions:

$$\Phi_{\tilde{P}} \equiv \lambda_+ \lambda_{\tilde{P}} - \frac{1}{8}\epsilon_{\tilde{P}\tilde{Q}\tilde{R}\tilde{S}\tilde{T}} \lambda_{QR} \lambda_{ST} = 0\,, \tag{2.17}$$

and hence $\lambda_{\tilde{P}}$'s are solved in terms of λ_+ and λ_{PQ}. Therefore the number of independent components of a pure spinor is 11 and together with all the other fields (including the conjugates to the independent components of λ) the entire system constitutes a free CFT with vanishing central charge.

The fact that the constrained cohomology of Q_B is in one to one correspondence with the light-cone degrees of freedom of superstring was shown in [15] using the SO(8) parametrization of a pure spinor. Besides being non-covariant, this parametrization is infinitely redundant and an indefinite number of supplementary ghosts had to be introduced. Nonetheless, subsequently the Lorentz invariance of the cohomology was demonstrated in [19].

The great advantage of this formalism is that one can compute the scattering amplitudes in a manifestly super-Poincaré covariant manner. For the massless modes, the physical unintegrated vertex operator is given by a simple form

$$U = \lambda^\alpha A_\alpha(x, \theta)\,, \tag{2.18}$$

where A_α is a spinor superfield satisfying the "on-shell" condition $(\gamma^{\mu_1\mu_2\cdots\mu_5})^{\alpha\beta} D_\alpha A_\beta = 0$ with $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\gamma^\mu\theta)_\alpha \frac{\partial}{\partial x^\mu}$. Then, with the pure spinor constraints, one easily verifies $Q_B U = 0$ and moreover finds that $\delta U = Q_B \Lambda$ represents the gauge transformation of

A_α. Its integrated counterpart $\int[dz]V(z)$, needed for calculation of n-point amplitudes with $n \geq 4$, is characterized by $Q_B V = \partial U$ and was constructed to be of the form [4, 5]

$$V = \partial\theta^\alpha A_\alpha + \Pi^\mu B_\mu + d_\alpha W^\alpha + \frac{1}{2} L^{\mu\nu}_{(\lambda)} F_{\mu\nu} \,. \tag{2.19}$$

Here, $B_\mu = (i/16)\gamma_\mu^{\alpha\beta} D_\alpha A_\beta$ is the gauge superfield, $W^\alpha = (i/20)(\gamma^\mu)^{\alpha\beta}(D_\beta B_\mu - \partial_\mu A_\beta)$ is the gaugino superfield, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength superfield and $L^{\mu\nu}_{(\lambda)}$ is the Lorentz generator for the pure spinor sector.

With these vertex operators, the scattering amplitude is expressed as $\mathcal{A} = \langle U_1(z_1)\, U_2(z_2)\, U_3(z_3) \int[dz_4]V_4(z_4)\cdots\int[dz_N]V_N(z_N)\rangle$, and can be computed in a covariant manner with certain rules assumed for the integration over the zero modes of λ^α and θ^α. The proposed prescription enjoys a number of required properties and leads to results which agree with those obtained in the RNS formalism [4],[12]–[14].

2.2 EPS formalism

Although the PS formalism briefly reviewed above has a number of remarkable features, for the reasons stated in the introduction, it is desirable to remove the PS constraints by extending the field space. Such an extension was achieved in a minimal manner in [29]. Skipping all the details, we give below the essence of the formalism.

Instead of the basic superinvariants forming the essentially second class algebra (2.6) $\sim$ (2.9), we introduce the four types of composite operators

$$j = \lambda^\alpha d_\alpha \,, \tag{2.20}$$
$$\mathcal{P}_P = \mathcal{N}_P^\mu \Pi_\mu \,, \tag{2.21}$$
$$\mathcal{R}_{PQ} = 2i\lambda_+^{-1}\mathcal{N}_P^\mu(\gamma_\mu\partial\theta)_Q \,, \tag{2.22}$$
$$\mathcal{S}_{PQ} = -(\partial\mathcal{N}_P^\mu)\mathcal{N}_Q^\mu \,, \tag{2.23}$$

where $\mathcal{N}_P^\mu$ are a set of five Lorentz vectors which are *null*, i.e. $\mathcal{N}_P^\mu \mathcal{N}_Q^\mu = 0$, defined by

$$\mathcal{N}_P^\mu \equiv -4\left(e_P^{+\mu} - \lambda_+^{-1}\lambda_{PQ} e_{\tilde{Q}}^{-\mu}\right) \,. \tag{2.24}$$

Note that j is the BRST-like current of Berkovits *now without PS constraints.* The virtue of this set of operators is that, by using the basic relations (2.6) $\sim$ (2.9), they can be shown to form a closed algebra which is of *first class*, namely without any central charges. This allows one to build a BRST-like nilpotent charge $\hat{Q}$ associated to this algebra. Introducing five sets of ferminonic ghost-anti-ghost pairs $(c_{\tilde{P}}, b_P)$ carrying conformal weights $(0,1)$ with the OPE

$$c_{\tilde{P}}(z) b_Q(w) = \frac{\delta_{PQ}}{z-w} \,, \tag{2.25}$$

and making use of the powerful scheme known as homological perturbation theory [35], $\hat{Q}$ is constructed as

$$\hat{Q} = \delta + Q + d_1 + d_2 \,, \tag{2.26}$$

where

$$\delta = -i \int [dz] b_P \Phi_{\tilde{P}} \,, \qquad Q = \int [dz] j \,, \tag{2.27}$$

$$d_1 = \int [dz] c_{\tilde{P}} \mathcal{P}_P \,, \qquad d_2 = -\frac{i}{2} \int [dz] c_{\tilde{P}} c_{\tilde{Q}} \mathcal{R}_{PQ} \,. \tag{2.28}$$

The operators (δ, Q, d_1, d_2) carry degrees $(-1, 0, 1, 2)$ under the grading $\deg(c_{\tilde{P}}) = 1$, $\deg(b_P) = -1$, $\deg(\text{rest}) = 0$ and the nilpotency of $\hat{Q}$ follows from the first class algebra mentioned above.

The crucial point of this construction is that by the main theorem of homological perturbation theory the cohomology of $\hat{Q}$ is guaranteed to be equivalent to that of Q with the constraint $\delta = 0$, i.e. with $\Phi_{\tilde{P}} = 0$, which are nothing but the PS constraints (2.17). Moreover, the underlying logic of this proof can be adapted to construct the massless vertex operators, both unintegrated and integrated, which are the generalization of the ones shown in (2.18) and (2.19) for the PS formalism. It should also be emphasized that in this formalism, due to the enlarged field space, one can construct the "B-ghost" field which realizes the important relation $\left\{\hat{Q}, B(z)\right\} = T^{\text{EPS}}(z)$.

To conclude this brief review, let us summarize the basic fields of the EPS formalism, their OPE's, the energy-momentum tensor $T^{\text{EPS}}(z)$ and the B-ghost field. Apart from the $(c_{\tilde{P}}, b_P)$ ghosts given in (2.25), the basic fields are the conjugate pairs $(\theta^\alpha, p_\alpha)$, $(\lambda^\alpha, \omega_\alpha)$, both of which carry conformal weights $(0, 1)$, and the string coordinate x^μ. Non-vanishing OPE's among them are

$$\theta^\alpha(z) p_\beta(w) = \frac{\delta^\alpha_\beta}{z - w} \,, \quad \lambda^\alpha(z) \omega_\beta(w) = \frac{\delta^\alpha_\beta}{z - w} \,, \quad x^\mu(z) x^\nu(w) = -\eta^{\mu\nu} \ln(z - w) \,. \tag{2.29}$$

The energy-momentum tensor is of the form

$$T^{\text{EPS}} = -\frac{1}{2} \partial x^\mu \partial x_\mu - p_\alpha \partial \theta^\alpha - \omega_\alpha \partial \lambda^\alpha - b_P \partial c_{\tilde{P}} \,, \tag{2.30}$$

with the total central charge vanishing. Finally, the B-ghost field is given by

$$B = -\omega_\alpha \partial \theta^\alpha + \frac{1}{2} b_P \Pi^-_{\tilde{P}} \,. \tag{2.31}$$

3. Operator mapping between BRST and light-cone formalisms for bosonic string

In order to gain insights into our main problem of connecting EPS with LCGS, we study in this section a simpler problem of mapping the BRST formalism into the light-cone formalism via a similarity transformation in bosonic string. Although the equivalence of these two formalisms was elucidated long ago in the seminal work of Kato and Ogawa [36], the explicit operator mapping constructed below, to our knowledge, is new.[3]

[3] A general discussion of a similarity transformation of the kind is given by W. Siegel in his web-published book "Fields".

3.1 From BRST to light-cone

Let us recall that the BRST charge Q for the bosonic string takes the form [36]

$$Q = \sum c_{-n} L_n - \frac{1}{2} \sum (m-n) : c_{-m} c_{-n} b_{m+n} : , \tag{3.1}$$

where the symbol : : stands for normal-ordering and the Virasoro generators are given by

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha^{\mu}_{n-k} \alpha_{\mu k} , \quad (n \neq 0) , \tag{3.2}$$

$$L_0 = \frac{1}{2} p^2 + \sum_{m \geq 1} \alpha^{\mu}_{-m} \alpha_{\mu m} - 1 . \tag{3.3}$$

The oscillators for the non-zero modes satisfy the commutation relations $[\alpha^{\mu}_m, \alpha^{\nu}_n] = m\eta^{\mu\nu}\,\delta_{m+n,0}$ with the metric convention $\eta_{\mu\nu} = (-,+,+,\ldots,+)$ and we have set $\alpha' = 1/2$. For any Lorentz vector A^{μ}, we define the light-cone components as $A^{\pm} \equiv \frac{1}{\sqrt{2}}(A^0 \pm A^{25})$ so that $A^{\mu}B_{\mu} = -(A^+B^- + A^-B^+) + A^iB^i \,, i = 1 \sim 24$. Then $\alpha^{\pm}_n$ satisfy the commutation relations $[\alpha^{\pm}_m, \alpha^{\mp}_n] = -m\delta_{m+n,0} \,, [\alpha^{\pm}_m, \alpha^{\pm}_n] = 0$.

In the light-cone formalism, the set of non-zero modes $(b_n, c_n, \alpha^+_n, \alpha^-_n)_{n \neq 0}$ are absent. In the BRST framework it means that they must form a "quartet" with respect to an appropriate nilpotent operator, to be called δ, and decouple from the cohomology of Q. Such a δ can easily be found in Q. When $p^+ \neq 0$, $\sum c_{-n}L_n$ contains the operator

$$\delta \equiv -p^+ \sum_{n \neq 0} c_{-n} \alpha^-_n , \tag{3.4}$$

and we have

$$\{\delta, \delta\} = 0 , \tag{3.5}$$

$$\left[\delta, \alpha^+_{-n}\right] = p^+ n c_{-n} , \qquad \{\delta, c_{-n}\} = 0 , \tag{3.6}$$

$$\{\delta, b_{-n}\} = -p^+ \alpha^-_{-n} , \qquad \left[\delta, \alpha^-_{-n}\right] = 0 . \tag{3.7}$$

This shows that indeed $(b_n, c_n, \alpha^+_n, \alpha^-_n)_{n \neq 0}$ form a quartet (i.e. two doublets) with respect to δ. (When $p^- \neq 0$, we may define a similar operator interchanging $\alpha^+_n \leftrightarrow \alpha^-_n, p^+ \leftrightarrow p^-$. In the following we consider the case $p^+ \neq 0$.)

Our aim is to demonstrate this decoupling in a manifest manner by constructing a similarity transformation

$$\bar{Q} = e^T Q e^{-T} , \tag{3.8}$$

where T is a suitable operator and

$$\bar{Q} = \delta + Q_{lc} , \tag{3.9}$$

$$Q_{lc} = c_0 L_0^{lc} , \tag{3.10}$$

$$L_0^{lc} = \frac{1}{2} p^2 + \sum_{n \geq 1} \alpha^i_{-n} \alpha^i_n - 1 . \tag{3.11}$$

As is almost evident from this form, the cohomology of $\bar{Q}$ consists of the usual light-cone on-shell states satisfying $L_0^{lc}|\psi\rangle = 0$ (modulo inessential double degeneracy due to c_0 mode). More precise discussion will be given at the end of this subsection.

To find such a transformation systematically, it is convenient to distinguish the members of the quartet by assigning non-vanishing degrees to them in such a way that (i) δ will carry degree -1 and (ii) the remaining part of Q will carry non-negative degrees. Such an assignment is given by[4]

$$\deg(\alpha_n^+) = 2\,, \qquad \deg(\alpha_n^-) = -2\,, \tag{3.12}$$

$$\deg(c_n) = 1\,, \qquad \deg(b_n) = -1\,, \qquad (n \neq 0) \tag{3.13}$$

$$\deg(\text{rest}) = 0\,. \tag{3.14}$$

Then, the BRST operator Q is decomposed according to this degree as

$$Q = \delta + Q_0 + d_1 + d_2 + d_3\,, \tag{3.15}$$

where the degree is indicated by the subscript, except for the δ carrying degree -1. The explicit forms of these operators are

$$\delta \equiv -p^+ \sum_{n\neq 0} c_{-n}\alpha_n^-\,, \tag{3.16}$$

$$Q_0 = c_0 L_0^{\text{tot}} = c_0 \left(L_0 + \sum_{n\neq 0} n : c_{-n} b_n : \right)\,, \tag{3.17}$$

$$d_1 = \sum_{n\neq 0} c_{-n}\tilde{L}_n - \frac{1}{2}\sum_{\text{nzm}} (m-n) : c_{-m}c_{-n}b_{m+n} :\,, \tag{3.18}$$

$$d_2 = -b_0 \sum_{n\neq 0} n : c_{-n}c_n :\,, \tag{3.19}$$

$$d_3 = -p^- \sum_{n\neq 0} c_{-n}\alpha_n^+\,, \tag{3.20}$$

where

$$\tilde{L}_n \equiv p^i\alpha_n^i + \frac{1}{2}\sum_{\text{nzm}} \alpha_{n-m}^\mu \alpha_{\mu m} \tag{3.21}$$

and "nzm" indicates that only the non-zero modes are to be summed.

The advantage of the decomposition above is that the nilpotency relation $\{Q, Q\} = 0$ splits into a set of simple relations at different degrees. Explicitly we have

$$(E_{-2}) \quad \{\delta, \delta\} = 0\,, \tag{3.22}$$

$$(E_{-1}) \quad \{\delta, Q_0\} = 0\,, \tag{3.23}$$

$$(E_0) \quad \{Q_0, Q_0\} + 2\{\delta, d_1\} = 0\,, \tag{3.24}$$

$$(E_1) \quad \{Q_0, d_1\} + \{\delta, d_2\} = 0\,, \tag{3.25}$$

[4] This grading is essentially a refined version of the one used in [36].

$$(E_2) \quad \{d_1, d_1\} + 2\{Q_0, d_2\} + 2\{\delta, d_3\} = 0\,, \tag{3.26}$$
$$(E_3) \quad \{Q_0, d_3\} + \{d_1, d_2\} = 0\,, \tag{3.27}$$
$$(E_4) \quad \{d_2, d_2\} + 2\{d_1, d_3\} = 0\,, \tag{3.28}$$
$$(E_5) \quad \{d_2, d_3\} = 0\,, \tag{3.29}$$
$$(E_6) \quad \{d_3, d_3\} = 0\,. \tag{3.30}$$

Using these relations, we now show that an operator R exists such that d_1, d_2 and d_3 can be removed by a similarity transformation in the following manner:

$$\begin{aligned} Q &= \delta + Q_0 + d_1 + d_2 + d_3 = e^{-R}(\delta + Q_0)e^R \\ &= \delta + Q_0 + [\delta, R] + [Q_0, R] + \frac{1}{2}[[\delta, R,]\, R] + \frac{1}{2}[[Q_0, R,]\, R] + \cdots\,. \end{aligned} \tag{3.31}$$

Since Q_0 is easily seen to be nilpotent, together with the relations (E_{-2}) and (E_{-1}) we see that $\delta + Q_0$ is nilpotent. This of course is necessary for the consistency of the relation (3.31), but it is non-trivial to prove that the series terminates at finite terms to precisely reproduce Q.

To find R and prove (3.31), knowledge of the homology of the operator δ will be useful. To this end, consider the operator $\hat{K}$ of degree 1 given by

$$\hat{K} \equiv \frac{1}{p^+} \sum_{n \neq 0} \frac{1}{n} \alpha^+_{-n} b_n\,. \tag{3.32}$$

Further define

$$\hat{N} \equiv \left\{\delta, \hat{K}\right\} = \sum_{n \neq 0} : \left(c_{-n} b_n - \frac{1}{n} \alpha^+_{-n} \alpha^-_n\right) : \,. \tag{3.33}$$

It is easy to see that $\hat{N}$ is an extension of the ghost number operator and assigns the "$\hat{N}$-number" $(1, -1, 1, -1)$ to the quartet $(c_n, b_n, \alpha^+_n, \alpha^-_n)$. Now let $\mathcal{O}$ be a δ-closed operator carrying $\hat{N}$-number n, i.e. $[\delta, \mathcal{O}\} = 0$ and $\left[\hat{N}, \mathcal{O}\right] = n\mathcal{O}$. If $n \neq 0$, we can express $\mathcal{O}$ as

$$\mathcal{O} = \frac{1}{n}\left[\{\delta, \hat{K}\}, \mathcal{O}\right] = \frac{1}{n}\left[\delta, \left\{\hat{K}, \mathcal{O}\right\}\right] \tag{3.34}$$

and hence $\mathcal{O}$ is δ-exact. So the non-trivial homology of δ can only be in the sector where $\hat{N} = 0$.

We are now ready for the construction of R. Consider first the relation (E_0). Since Q_0 itself is nilpotent, it dictates $\{\delta, d_1\} = 0$, namely that d_1 is δ-closed. But since $\left[\hat{N}, d_1\right] = d_1$, application of (3.34) allows us to write

$$d_1 = [\delta, R_2]\,, \qquad R_2 \equiv \left\{\hat{K}, d_1\right\}\,. \tag{3.35}$$

Next, look at the relation (E_1). Since $\{\delta, d_2\} = 0$ holds by inspection, (E_1) actually splits into two relations:

$$\{\delta, d_2\} = 0\,, \tag{3.36}$$
$$\{Q_0, d_1\} = 0\,. \tag{3.37}$$

Now since $\left[\hat{N}, d_2\right] = 2d_2$, the former relation together with (3.34) tells us that d_2 can be written as

$$d_2 = [\delta, R_3] \,, \qquad R_3 \equiv \frac{1}{2}\left\{\hat{K}, d_2\right\} . \tag{3.38}$$

Comparing with the expansion (3.31), the results obtained above suggest that R is given by $R = R_2 + R_3$. Indeed the rest of the work is to confirm this expectation. The main points are

- d_3 is produced as $d_3 = \frac{1}{2}\left[[\delta, R_2], R_2\right] + [Q_0, R_3]$
- All the unwanted commutators indeed vanish.

Since the details are somewhat long, we relegate them to appendix B.1. To summarize, we have so far shown

$$Q = e^{-R}\tilde{Q}e^{R} \,, \qquad \tilde{Q} = \delta + Q_0 \,, \tag{3.39}$$

$$R = \left\{\hat{K}, d_1 + \frac{1}{2}d_2\right\} . \tag{3.40}$$

Next, we shall further reduce $\tilde{Q}$ to $\bar{Q}$ so that the light-cone structure is manifestly visible. To this end, define an operator $\tilde{K}$, similar to but different from $\hat{K}$, by

$$\tilde{K} \equiv \frac{1}{p^+}\sum_{n\neq 0} \alpha^+_{-n} b_n \,. \tag{3.41}$$

Then the anti-commutator with δ yields the operator

$$\tilde{N} = \left\{\delta, \tilde{K}\right\} = \sum_{n\geq 1}\left(-\alpha^-_{-n}\alpha^+_n - \alpha^+_{-n}\alpha^-_n + n(b_{-n}c_n + c_{-n}b_n)\right) , \tag{3.42}$$

which evidently measures the Virasoro level of the member of the quartet $q_{-n} = (\alpha^{\pm}_{-n}, b_{-n}, c_{-n})$ in the sense $\left[\tilde{N}, q_{-n}\right] = nq_{-n}$. Clearly $\tilde{N}$ commutes with $\tilde{Q}$. Further it is easy to check that, just like $\hat{K}$, the operator $\tilde{K}$ anti-commutes with Q_0. Therefore (3.42) gives the relation

$$\left\{\tilde{Q}, \tilde{K}\right\} = \tilde{N} \,. \tag{3.43}$$

Now note that Q_0 given in (3.17) can be decomposed as

$$Q_0 = c_0 L_0^{lc} + c_0\tilde{N} = Q_{lc} + c_0\tilde{N} \,, \tag{3.44}$$

where L_0^{lc} and Q_{lc} were defined in (3.11) and (3.10). Therefore an additional similarity transformation by $S \equiv -c_0\tilde{K}$ precisely removes the $c_0\tilde{N}$ part of Q_0 and achieves our goal:

$$\begin{aligned}\bar{Q} &\equiv e^{T}Qe^{-T} = e^{S}e^{R}Qe^{-R}e^{-S} \\ &= e^{-c_0\tilde{K}}\tilde{Q}e^{c_0\tilde{K}} = \tilde{Q} - c_0\left\{\tilde{K}, \tilde{Q}\right\} = \tilde{Q} - c_0\tilde{N} = \delta + Q_{lc} \,. \end{aligned}\tag{3.45}$$

We now give a rather standard argument to confirm that $\bar{Q}$ defines the light-cone theory. Let $|\psi\rangle$ be a $\bar{Q}$-closed state at the Virasoro level n with respect to the quartet oscillators, i.e. $\bar{Q}|\psi\rangle = 0$ and $\tilde{N}|\psi\rangle = n|\psi\rangle$. Then, since $\left\{\bar{Q}, \tilde{K}\right\} = \tilde{N}$ holds, $n|\psi\rangle = \tilde{N}|\psi\rangle = \left\{\bar{Q}, \tilde{K}\right\}|\psi\rangle = \bar{Q}(\tilde{K}|\psi\rangle)$. Hence $|\psi\rangle$ with $n \neq 0$, i.e. with quartet excitations, is cohomologically trivial. Disregarding such states, the cohomology of $\bar{Q}$ is reduced to that of Q_{lc}. Now in the reduced space a general state $|\psi\rangle$ can be written as $|\psi\rangle = |\phi\rangle + c_0|\chi\rangle$, where $|\phi\rangle$ and $|\chi\rangle$ do not contain ghost zero modes. Q_{lc}-closed condition on $|\psi\rangle$ imposes the light-cone on-shell condition $L_0^{lc}|\phi\rangle = 0$ on $|\phi\rangle$. On the other hand, using $\{Q_{lc}, b_0\} = L_0^{lc}$ we can make a similar argument as above to conclude that if L_0^{lc} does not annihilate $c_0|\chi\rangle$, it can actually be written as a Q_{lc}-exact state. Thus the cohomology of Q_{lc} is represented by $|\phi\rangle + c_0|\chi\rangle$ with $L_0^{lc}|\phi\rangle = 0$ and $L_0^{lc}|\chi\rangle = 0$. Apparently the spectrum is doubly degenerate but, as is well-known, $|\chi\rangle$ does not contribute to the physical amplitude. Indeed the inner product of two physical states $|\psi\rangle$ and $|\psi'\rangle$ is given by $\langle\psi'|c_0|\psi\rangle = \langle\phi'|c_0|\phi\rangle = \langle\phi'|\phi\rangle_{lc}$, where the subscript "$lc$" signifies the space without the ghost zero modes, and computations in the light-cone theory are reproduced.

3.2 DDF operators and Virasoro Algebra

Having succeeded in connecting the covariant BRST and the light-cone formalisms by an explicit similarity transformation, it is natural and interesting to ask how various operators on both sides are mapped by this transformation.

One immediate question is: what is the counterpart of the transverse oscillators α_n^i on the covariant BRST side ? Since such operators must satisfy the same commutation relations as α_n^i's, it is natural to guess that they must be intimately related to the DDF operators [34].

Let us write the mode expansion of the open string coordinate $X^\mu(\sigma,\tau)$ at $\sigma = 0$ as

$$X^\mu(\tau) = x^\mu + p^\mu\tau + iY^\mu(\tau)\,, \tag{3.46}$$

$$Y^\mu(\tau) = \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\,, \tag{3.47}$$

where $0 \le \tau \le 2\pi$. Then, the DDF operator A_n^i is constructed from the photon vertex operator at a special light-like momentum as

$$A_n^i = e^{inx^+/p^+}\check{A}_n^i\,, \tag{3.48}$$

$$\check{A}_n^i \equiv \int [d\tau] e^{in\tau}\dot{X}^i(\tau)e^{-(n/p^+)Y^+(\tau)}\,, \tag{3.49}$$

where $\int[d\tau] \equiv \int_0^{2\pi} d\tau/2\pi$ and for convenience we have separated the zero-mode phase factor. Expanding $\check{A}_n^i$ out a few terms in powers of $1/p^+$ and noting that the similarity transformation by $-S = c_0\tilde{K}$ does not affect α_n^i, one is lead to conjecture the simple relation

$$\check{A}_n^i = e^{-T}\alpha_n^i e^T = e^{-R}\alpha_n^i e^R = \alpha_n^i - \left[R,\alpha_n^i\right] + \frac{1}{2}\left[R,\left[R,\alpha_n^i\right]\right] + \cdots\,. \tag{3.50}$$

To prove it, we need to know the explicit form of R defined in (3.40). After a straightforward calculation we get

$$R = R_2 + R_3 = \frac{1}{p^+}\sum_{k\neq 0}\frac{1}{k}\alpha_{-k}^+\tilde{L}_k^{\rm tot}\,, \tag{3.51}$$

$$\tilde{L}_k^{\rm tot} \equiv \tilde{L}_k + \sum_{n\neq 0} nc_{-n}b_{k+n}\,, \tag{3.52}$$

where $\tilde{L}_k$ was defined in (3.21). It is easy to show that, with respect to $\tilde{L}_n^{\rm tot}$ operators, $\dot{Y}^i(\tau), \dot{Y}^+(\tau)$ and $b(\tau) = \sum_n b_n e^{-in\tau}$ behave as conformal primary fields of dimension 1, while $c(\tau) = \sum_n c_n e^{-in\tau}$ behaves as one of dimension 0. Namely we have

$$\left[\tilde{L}_k^{\rm tot}, \phi(\tau)\right] = e^{ik\tau}\left(\frac{1}{i}\partial_\tau + k\right)\phi(\tau)\,, \tag{3.53}$$

$$\phi(\tau) = \left(\dot{Y}^i(\tau), \dot{Y}^+(\tau), b(\tau)\right)\,, \tag{3.54}$$

$$\left[\tilde{L}_k^{\rm tot}, c(\tau)\right] = e^{ik\tau}\frac{1}{i}\partial_\tau c(\tau)\,. \tag{3.55}$$

Using these properties, the action of R on the basic fields is worked out as

$$\left[R, \dot{Y}^i(\tau)\right] = -\frac{1}{p^+}\frac{1}{i}\partial_\tau\left(Y^+(\tau)\dot{Y}^i(\tau)\right)\,, \tag{3.56}$$

$$\left[R, Y^+(\tau)\right] = -\frac{1}{p^+}\frac{1}{i}Y^+(\tau)\partial_\tau Y^+(\tau)\,, \tag{3.57}$$

$$\left[R, b(\tau)\right] = -\frac{1}{p^+}\frac{1}{i}\partial_\tau\left(Y^+(\tau)b(\tau)\right)\,, \tag{3.58}$$

$$\left[R, c(\tau)\right] = -\frac{1}{p^+}\frac{1}{i}Y^+(\tau)\partial_\tau c(\tau)\,. \tag{3.59}$$

Now the computation of the inverse similarity transformation is straightforward: The single commutator is given by

$$\left[R, \alpha_n^i\right] = \int [d\tau] e^{in\tau}\left[R, \dot{Y}^i(\tau)\right] = \frac{n}{p^+}\int [d\tau] e^{in\tau} Y^+(\tau)\dot{Y}^i(\tau)\,. \tag{3.60}$$

Further the double commutator yields

$$\left[R, \left[R, \alpha_n^i\right]\right] = \frac{n}{p^+}\int [d\tau] e^{in\tau}\left[R, Y^+(\tau)\dot{Y}^i(\tau)\right] = \left(\frac{n}{p^+}\right)^2\int [d\tau] e^{in\tau} Y^+(\tau)^2\dot{Y}^i(\tau)\,. \tag{3.61}$$

It should be clear that this process exponentiates the factor $(n/p^+)Y^+(\tau)$ and the relation (3.50) is proved.

The essential property of the DDF operators is that they commute with the Virasoro and the BRST operators. This can be understood using our similarity transformation as follows. Since A_n^i does not contain (b, c) ghosts, we only need to show $\left[Q, A_n^i\right] = 0$. By the similarity transformation, this is mapped to

$$\left[\delta + Q_{lc}, e^{inx^+/p^+}\alpha_n^i\right]\,. \tag{3.62}$$

δ trivially commutes with $e^{inx^+/p^+}\alpha_n^i$. Q_{lc} also commutes with it since

$$\begin{aligned}\left[L_0^{lc}, e^{inx^+/p^+}\alpha_n^i\right] &= \left[L_0^{lc}, e^{inx^+/p^+}\right]\alpha_n^i + e^{inx^+/p^+}\left[L_0^{lc}, \alpha_n^i\right] \\ &= ne^{inx^+/p^+}\alpha_n^i - ne^{inx^+/p^+}\alpha_n^i = 0\,. \end{aligned}\tag{3.63}$$

Note that the phase factor in front is crucial for the cancellation.

A related question of interest is: what happen to the Virasoro generators when transformed to the light-cone side? Consider the total Virasoro operators $L_n^{\rm tot}$ including the ghost part. It is convenient to represent it as $L_n^{\rm tot} = \{Q, b_n\}$ since we already know how Q is similarity-transformed. Therefore the Virasoro operators on the light-cone side can be computed as

$$\bar{L}_n^{\rm tot} = e^S e^R L_n^{\rm tot} e^{-R} e^{-S} = \{\bar{Q}, \bar{b}_n\} \,, \tag{3.64}$$

$$\bar{b}_n = e^S e^R b_n e^{-R} e^{-S} \,. \tag{3.65}$$

Transformation of b_n by e^R can be performed in a manner similar to that for α_n^i, with the result

$$\tilde{b}_n \equiv e^R b_n e^{-R} = \int [d\tau] e^{in\tau} e^{(n/p^+)Y^+(\tau)} b(\tau) \,. \tag{3.66}$$

Further transformation by e^S acts only on the zero mode part of $b(\tau)$ in the integrand and gives

$$\bar{b}_n = \int [d\tau] e^{in\tau} e^{(n/p^+)Y^+(\tau)} (b(\tau) + \tilde{K}) \,. \tag{3.67}$$

Then by a straightforward calculation we obtain

$$\bar{L}_n^{\rm tot} = \int [d\tau] e^{in\tau} e^{(n/p^+)Y^+(\tau)} \left(-n\check{c}(\tau)(b(\tau) + \tilde{K}) - ip^+\dot{Y}^-(\tau) + \tilde{N} + L_0^{lc}\right) \,, \tag{3.68}$$

where $\check{c}(\tau) \equiv c(\tau) - c_0$. Note that this is of the form $B_n + C_n L_0^{lc}$, where B_n and C_n consist entirely of unphysical fields. Since L_0^{lc} annihilates the physical states, Virasoro operators are essentially inert in the light-cone formalism. One may say that the gauge transformations are already "used up" in the light-cone formulation or they only act on the unphysical sector.

4. Relating EPS and light-cone GS formalisms by similarity transformation

We now make use of the experience gained in the previous section to construct the similarity transformation connecting the light-cone GS and the EPS formalisms for superstring. Since the construction is more involved compared to the bosonic string case, we shall present it in two steps so that our strategy will be transparent. Conventions, to be used below, for Γ^μ matrices, spinors and their U(5) and SO(8) parametrizations are explained in appendix A.

4.1 Similarity transformation I

Just as in the bosonic case, the final BRST operator on the LCGS side should be of the form

$$\bar{Q} = \delta_{lc} + Q_{lc} \,, \tag{4.1}$$

$$Q_{lc} = c_0 L_0^{lc} = c_0 \left(L_{0,B}^{lc} + L_{0,F}^{lc}\right) . \tag{4.2}$$

Here $L^{lc}_{0,B}$ is the bosonic part of the on-shell operator identical to (3.11), while $L^{lc}_{0,F}$ is the fermionic part given by

$$L^{lc}_{0,F} = \sum_{n\geq 1} n S^a_{-n} S^a_n \,, \tag{4.3}$$

where S^a_n $(a = 1 \sim 8)$ are the SO(8)-chiral spinor oscillators expressing the physical fermionic excitations of the light-cone GS string. They should satisfy the self-conjugate anti-commutation relations

$$\left\{ S^a_m, S^b_n \right\} = \delta^{ab} \delta_{m,-n} \,. \tag{4.4}$$

δ_{lc} in (4.1) is the operator consisting of "unphysical" oscillators, which allows us to identify and drop the unwanted states from the cohomology. Thus our first task is to identify S^a_n and construct appropriate δ_{lc} in terms of the EPS variables.

The basic fermionic spinor fields of EPS are the conjugate pairs $(\theta^\alpha, p_\alpha)$, which can be decomposed into SO(8) chiral and anti-chiral pairs as $(\theta^a, p_a), (\theta^{\dot a}, p_{\dot a})$. From the SO(8)-chiral pair, one can construct two self-conjugate fields as

$$S^a \equiv \frac{1}{\sqrt{2p^+}} \left(p^a + p^+ \theta^a \right) \,, \qquad \tilde{S}^a \equiv \frac{1}{\sqrt{2p^+}} \left(p^a - p^+ \theta^a \right) \,. \tag{4.5}$$

Their modes satisfy the anti-commutation relations

$$\left\{ S^a_m, S^b_n \right\} = \delta^{ab} \delta_{m,-n} \,, \qquad \left\{ \tilde{S}^a_m, \tilde{S}^b_n \right\} = -\delta^{ab} \delta_{m,-n} \,. \tag{4.6}$$

Thus, it is natural to identify S^a in (4.5) as the physical fermionic field of the GS formalism while the negatively-normed $\tilde{S}^a$ operator, together with the anti-chiral pair $(\theta^{\dot a}, p_{\dot a})$, should be regarded as unphysical.

There are also five pairs of fermionic ghosts $(b_P, c_{\tilde P})_{P=1\sim 5}$ in the EPS formalism, which will be divided into $(b_p, c_{\tilde p})_{p=1\sim 4}$ and $(b_5, c_{\tilde 5})$. Since the light-cone fields $\partial x^\pm$ should be removed together with a "(b,c)" ghost pair just as in the bosonic string, it is natural to make the identification $(b,c) = (\xi b_5, \xi^{-1} c_{\tilde 5})$, where the rescaling factor ξ will be appropriately chosen later.

We can now depict the correspondence between the degrees of freedom of light-cone GS and EPS by the following diagram:

$$\partial x^i \oplus \underbrace{\partial x^\pm \oplus (b,c)}_{\text{quartet}} \,, \tag{4.7}$$

$$\mathrlap{\overbrace{\phantom{S^a \oplus \tilde{S}^a \oplus (\theta^{\dot a}, p_{\dot a})}}^{(\theta^\alpha, p_\alpha)}} S^a \oplus \underbrace{\tilde{S}^a \oplus (\theta^{\dot a}, p_{\dot a}) \oplus (b_p, c_{\tilde p}) \oplus (\lambda^\alpha, \omega_\alpha) \,.}_{\text{quartets}} \tag{4.8}$$

The structure in the upper line is precisely that of the bosonic string. In the lower line, one sees that, as far as the counting of the degrees of freedom is concerned, the bosonic fields $(\lambda^\alpha, \omega_\alpha)$ and the fermionic fields $(\tilde{S}^a, \theta^{\dot a}, p_{\dot a}, b_p, c_{\tilde p})$ can form 16 unphysical quartets so that only S^a will be left as physical, as we wish.

How the quartets are formed is dictated by the structure of the operator δ_{lc}. It is clear that to remove the quartet $(\partial x^\pm, b, c)$ we may use the same structure as in the bosonic

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case, namely $\delta_b \equiv -p^+ \sum_{n\neq 0} c_{-n}\alpha_n^-$. It should also be evident that the sets $(\lambda^{\dot a}, \omega_{\dot a}, \theta^{\dot a}, p_{\dot a})$ consisting of SO(8) conjugate spinors form eight quartets with respect to the nilpotent operator $\delta_c \equiv \sum_n \lambda^{\dot a}_{-n} p_{\dot a,n}$. Indeed, we have

$$\begin{aligned} \{\delta_c, \theta_n^{\dot a}\} &= \lambda_n^{\dot a}\,, & [\delta_c, \lambda_n^{\dot a}] &= 0\,, \\ [\delta_c, \omega_{\dot a,n}] &= p_{\dot a,n}\,, & \{\delta_c, p_{\dot a,n}\} &= 0\,. \end{aligned} \tag{4.9}$$

We are still left with the remaining unphysical fields $(\tilde S^a, b_p, c_{\tilde p}, \lambda^a, \omega_a)$ to deal with. The most natural way to form quartets is to first construct, out of $\tilde S^a$ and $(b_p, c_{\tilde p})$, eight conjugate pairs $(\tilde S^{+a}, \tilde S^{-a})$ satisfying the anti-commutation relations

$$\left\{\tilde S_m^{+a}, \tilde S_n^{+b}\right\} = \left\{\tilde S_m^{-a}, \tilde S_n^{-b}\right\} = 0\,, \qquad \left\{\tilde S_m^{+a}, \tilde S_n^{-b}\right\} = \delta^{ab}\delta_{m+n,0}\,. \tag{4.10}$$

Then, the fields $(\tilde S^{+a}, \tilde S^{-a}, \lambda^a, \omega_a)$ form eight quartets with respect to the operator $\delta_s \equiv -2\sum_n \lambda^a_{-n}\tilde S_n^{-a}$. (The factor of -2 in front is for later convenience.)

To construct $(\tilde S^{+a}, \tilde S^{-a})$, we first reshuffle $(b_p, c_{\tilde p})$ to form self-conjugate fields g_a in the following way. Let $u_{\dot a}$ be a constant conjugate spinor normalized as $u_{\dot a}u_{\dot a} = 2$ and form

$$g_a \equiv \sqrt{p^+}\, e_p^{+i}\gamma^i_{a\dot a}u_{\dot a}c_{\tilde p} + \frac{1}{\sqrt{p^+}} e_{\tilde p}^{-i}\gamma^i_{a\dot a}u_{\dot a}b_p\,. \tag{4.11}$$

Here $\gamma^i_{a\dot a}$ are the SO(8) Pauli matrices and $(e_p^{+i}, e_{\tilde p}^{-i})$ are the U(4) projectors defined just like the U(5) counterparts given in (2.13). Then from the anti-commutation relation $\{b_p, c_{\tilde q}\} = \delta_{pq}$, one can check that g_a is self-conjugate, i.e. $\{g_a, g_b\} = \delta_{ab}$. The definition (4.11) can be inverted to yield

$$b_p = \sqrt{p^+}\, e_p^{+i}(\gamma^i u)_a g_a\,, \qquad c_{\tilde p} = \frac{1}{\sqrt{p^+}} e_{\tilde p}^{-i}\left(\gamma^i u\right)_a g_a\,. \tag{4.12}$$

Furthermore, one can easily prove the relation

$$g_{a,-n}g_{a,n} = b_{p,-n}c_{\tilde p,n} + c_{\tilde p,-n}b_{p,n}\,, \tag{4.13}$$

which holds regardless of the explicit choice of $u_{\dot a}$. Using g_a we now define $(\tilde S_n^{+a}, \tilde S_n^{-a})$ as

$$\tilde S_n^{+a} \equiv \frac{1}{\sqrt{2p^+}}\left(\tilde S_n^a + g_n^a\right)\,, \qquad \tilde S_n^{-a} \equiv \sqrt{\frac{p^+}{2}}\left(-\tilde S_n^a + g_n^a\right)\,. \tag{4.14}$$

Then these oscillators correctly satisfy the anti-commutation relations (4.10).

Summarizing, the appropriate δ_{lc} is given by

$$\delta_{lc} = \delta_b + \delta_c + \delta_s\,, \tag{4.15}$$

$$\delta_b = -p^+\sum_{n\neq 0} c_{-n}\alpha_n^- = -i\int [dz]p^+(c(z)\partial x^-(z))_{\rm nzm}\,, \tag{4.16}$$

$$\delta_c = \sum_n \lambda^{\dot a}_{-n}p_{\dot a,n} = \int [dz]\lambda^{\dot a}(z)p_{\dot a}(z)\,, \tag{4.17}$$

$$\delta_s = -2\sum_n \lambda^a_{-n}\tilde S_n^{-a} = -2\int [dz]\lambda^a(z)\tilde S^{-a}(z)\,. \tag{4.18}$$

where the subscript "nzm" means omitting the zero mode part. Just as in the bosonic string case, we can easily construct the homotopy operator K with respect to δ_{lc} such that $\{\delta_{lc}, K\} = N$, where N is the level-counting operator for the unphysical modes. It is given by

$$K = \frac{1}{p^+}\sum_{n\neq 0} \alpha^+_{-n} b_n + \sum_{n\neq 0} n\left(\theta^{\dot{a}}_n \omega_{\dot{a},-n} - \frac{1}{2}\tilde{S}^{+a}_n \omega_{a,-n}\right) . \tag{4.19}$$

We may now follow the process employed in (3.45) backwards and add the operator N to $\bar{Q}$, to get a "more covariantized" form. Using the definition of $\tilde{S}^{\pm a}$ as well as the relation (4.13), we get

$$\bar{Q}' \equiv e^{c_0 K}\bar{Q}e^{-c_0 K} = \bar{Q} + N = \delta_{lc} + Q_0 \,, \tag{4.20}$$

$$Q_0 = c_0 L_0^{\mathrm{tot}} \,, \tag{4.21}$$

$$\begin{aligned} L_0^{\mathrm{tot}} &= \frac{1}{2}p^\mu p_\mu + \sum_{n\geq 1}\alpha^\mu_{-n}\alpha_{\mu n} + \\ &+ \sum_{n\geq 1} n\Big(p_{\alpha,-n}\theta^\alpha_n + \theta^\alpha_{-n}p_{\alpha,n} + \lambda^\alpha_{-n}\omega_{\alpha,n} - \omega_{\alpha,-n}\lambda^\alpha_n + c_{\tilde{P},-n}b_{P,n} + b_{P,-n}c_{\tilde{P},n}\Big) . \end{aligned} \tag{4.22}$$

It is important to note that L_0^{tot} above is precisely the "zero mode part" of the energy-momentum tensor T^{EPS} of the EPS formalism given in (2.30). Thus, from now on we shall denote it by T_0^{EPS}.

Before we describe the main part of the similarity transformation, let us make one further manipulation to bring $\bar{Q}'$ to a yet better form. Let us write out δ_s more explicitly. It reads

$$\delta_s = -2\int [dz]\lambda^a(z)\tilde{S}^{-a}(z) = \int [dz]\left(\lambda^a p_a - p^+\lambda^a\theta_a - \sqrt{2p^+}\,\lambda^a g_a\right) . \tag{4.23}$$

A non-trivial fact is that this is produced from $\int [dz]\lambda^a p_a$ by the following simple similarity transformation, as can be readily verified:

$$\delta_s = e^T \int [dz]\lambda^a p_a e^{-T} \,, \tag{4.24}$$

$$T = \sqrt{2p^+}\int [dz]\theta^a g_a \,. \tag{4.25}$$

Since T commutes with the rest of $\bar{Q}'$, we can now combine $\lambda^{\dot{a}}p_{\dot{a}}$ in δ_c and $\lambda^a p_a$ above to produce the covariant form $\lambda^\alpha p_\alpha$. In this way we get

$$\tilde{Q} \equiv e^{-T}\bar{Q}'e^T = \int [dz]\left(\lambda^\alpha p_\alpha - ip^+\left(c\partial x^-\right)_{nzm}\right) + c_0 T_0^{\mathrm{EPS}} \,. \tag{4.26}$$

Notice that the first term $\lambda^\alpha p_\alpha$ is precisely the combination that appears in $\lambda^\alpha d_\alpha$, the main part of the BRST-like operator $\hat{Q}$ for EPS. Thus the remaining task is to construct the rest of the similarity transformation which converts $\hat{Q}$ to $\tilde{Q}$.

4.2 Similarity transformation II

Recall the form of $\hat{Q}$:

$$\begin{aligned}\hat{Q} &= \delta + Q + d_1 + d_2 \\ &= -ib_P\Phi_{\tilde{P}} + \lambda^\alpha d_\alpha + c_{\tilde{P}}\mathcal{P}_P - \frac{i}{2}c_{\tilde{P}}c_{\tilde{Q}}\mathcal{R}_{PQ}\,. \end{aligned} \tag{4.27}$$

Here and hereafter, we shall omit the integral symbol $\int[dz]$. Comparing it to $\tilde{Q}$ in (4.26), one observes the following:

- The first term $\delta = -ib_P\Phi_{\tilde{P}}$ and the last term $d_2 = -\frac{i}{2}c_{\tilde{P}}c_{\tilde{Q}}\mathcal{R}_{PQ}$ must be removed completely.
- $c_{\tilde{5}}\mathcal{P}_5$ part of $d_1 = c_{\tilde{P}}\mathcal{P}_P$ contains the structure $\sim (c\partial x^-)_{\rm nzm}$ appearing in $\tilde{Q}$. We must remove the rest, including the zero mode part $(c\partial x^-)_0$.
- The structure $c_0T_0^{\rm EPS}$ must be generated.
- As for $Q = \lambda^\alpha d_\alpha$, we must keep the $\lambda^\alpha p_\alpha$ part and remove the rest.

Below we will perform these operations by a series of similarity transformations.

The first step is to remove δ by a similarity transformation of the form $e^X\hat{Q}e^{-X}$. We can achieve this by an operator X with the property $XQ = -\delta$. It is given by

$$X = -ib_PX_{\tilde{P}}\,, \qquad X_{\tilde{P}} = \bar{\mathcal{N}}^\mu_{\tilde{P}}\theta\gamma_\mu\lambda\,. \tag{4.28}$$

Apart form shifting Q by $-\delta$, this similarity transformation translates $c_{\tilde{P}}$ by $-iX_{\tilde{P}}$, as is clear from the structure of X. In this way we get

$$Q^{[1]} \equiv e^X\hat{Q}e^{-X} = Q + (c_{\tilde{P}} - iX_{\tilde{P}})\mathcal{P}_P - \frac{i}{2}(c_{\tilde{P}} - iX_{\tilde{P}})(c_{\tilde{Q}} - iX_{\tilde{Q}})\mathcal{R}_{PQ}\,. \tag{4.29}$$

As the second step, we remove $c_{\tilde{p}}\mathcal{P}_p$. This can be done by a similarity transformation $e^YQ^{[1]}e^{-Y}$ with

$$Y = -2i\lambda_+^{-1}c_{\tilde{p}}d_p\,, \tag{4.30}$$

which, apart from a factor of 2, is very similar to the transformation used in [30] to connect $\hat{Q}$ with the Berkovits' BRST-like operator Q_B. Although rather tedious, the calculation is straightforward. It turns out that the net result is simply to remove $c_{\tilde{p}}$ from $Q^{[1]}$. Thus we obtain

$$Q^{[2]} \equiv e^YQ^{[1]}e^{-Y} = Q + c_{\tilde{5}}(\mathcal{P}_5 - X_{\tilde{p}}\mathcal{R}_{5p}) - iX_{\tilde{P}}\mathcal{P}_P + \frac{i}{2}X_{\tilde{P}}X_{\tilde{Q}}\mathcal{R}_{PQ}\,. \tag{4.31}$$

At this stage, let us try to generate the $c_0T_0^{\rm EPS}$ structure appearing in (4.26). This step was not needed in the bosonic string case, since the BRST operator already contained such a term. On the other hand, in the EPS (and PS) formalism, Virasoro generators are not present in $\hat{Q}$ and they can only appear through the fundamental relation $\left\{\hat{Q}, B(z)\right\} = T^{\rm EPS}(z)$. To construct the similarity transformation that produces $c_0T_0^{\rm EPS}$, we need to specify the precise definition of (b,c) ghosts in terms of $(b_5, c_{\tilde{5}})$. As already mentioned, $c_{\tilde{5}}\mathcal{P}_5$ term contains the structure $\sim c\partial x^-$ in the form $-4c_{\tilde{5}}\partial x_5^+$. For this to be equal to $-ip^+c\partial x^-$, with our convention $\partial x_5^+ = e_5^{+\mu}\partial x_\mu = -\partial x^-/\sqrt{2}$, the identification should be

$$c_{\tilde{5}} = -\frac{i}{2\sqrt{2}}p^+c\,, \qquad b_5 = \frac{i2\sqrt{2}}{p^+}b\,. \tag{4.32}$$

A characteristic property of the energy-momentum tensor is that it is unchanged under any similarity transformation $e^U T e^{-U}$ as long as the total Virasoro level of the exponent U is zero. Since all the similarity transformations we have performed are of this type, we have $\left\{Q^{[2]}, B^{[2]}(z)\right\} = T^{\mathrm{EPS}}(z)$, where $B^{[2]}(z)$ is the similarity-transform of the B-ghost operator at this stage. Then, using its zero mode $B_0^{[2]} = \int[dz]z^2B^{[2]}(z)$, a similarity transformation with the exponent $c_0B_0^{[2]}$ would produce $c_0T_0^{\mathrm{EPS}}$. This, however, is not quite the correct procedure. As $B_0^{[2]}$ contains b_0 mode, the exponent has a term $\sim c_0b_0$ and the similarity transformation produces an infinite series. Thus, we must first subtract off the $b_0(\propto b_{5,0})$-dependent part from $B_0^{[2]}$. The original B field given in (2.31) contains the b_5-dependent part

$$\frac{1}{2\sqrt{2}}b_5(\partial x^+ - i\sqrt{2}\theta^a\partial\theta^a)\,. \tag{4.33}$$

The first similarity transformation by e^T does not modify this structure. On the other hand, the second transformation by e^X adds the contribution

$$X(-\omega_a\partial\theta^a) = \frac{i}{2}b_5\theta^a\partial\theta^a\,, \tag{4.34}$$

which is seen to precisely cancel the second term of (4.33). Since the subsequent transformation by e^Y does not produce any new term containing b_5, we find that the b_0-dependent part of $B_0^{[2]}$ is given by $(1/2\sqrt{2})b_{5,0}(\partial x^+)_0$, which according to the identification (4.32) is exactly equal to b_0. Therefore, the appropriate transformation is of the form e^Z with

$$Z = c_0\check{B}_0\,, \qquad \check{B}_0 \equiv B_0^{[2]} - b_0\,. \tag{4.35}$$

A simple calculation then yields

$$Q^{[3]} \equiv e^ZQ^{[2]}e^{-Z} = Q^{[2]} + c_0T_0^{\mathrm{EPS}} - c_{5,0}(\mathcal{P}_5 - X_{\tilde{p}}\mathcal{R}_{5p})_0\,. \tag{4.36}$$

Note that apart from the desired term $c_0T_0^{\mathrm{EPS}}$, the above transformation also generated the last term, which comes from $-c_0\left\{b_0, Q^{[2]}\right\} = -c_{\tilde{5},0}\left\{b_{5,0}, Q^{[2]}\right\}$. It subtracts off from the term $c_5(\mathcal{P}_5 - X_{\tilde{p}}\mathcal{R}_{5p})$ in $Q^{[2]}$ the part proportional to $c_{5,0}$. This turns out to be quite significant. As already pointed out at the beginning of this subsection, the zero mode part of the term $-4c_{\tilde{5}}\partial x_5^+ = -ip^+c\partial x^-$ in $c_{\tilde{5}}\mathcal{P}_5$ must be removed in order to obtain the correct structure we need in $\tilde{Q}$ of (4.26). What we have found is that the subtraction of b_0 from $B_0^{[2]}$ needed to correctly generate the $c_0T_0^{\mathrm{EPS}}$ structure simultaneously does this job for us. This is a subtle but compelling indication that our similarity transformation is on the right track.

What remains to be done is to remove the rest of the unwanted fields by judicious similarity transformations and bring $Q^{[3]}$ into $\tilde{Q}$ given in (4.26). We will achieve this in two steps.

Consider first the decoupling of $(\lambda_{p5}, \omega_{\tilde{p}\tilde{5}}, \theta_{p5}, p_{\tilde{p}\tilde{5}})$, which form a quartet with respect to the operator $\delta^{[3]} \equiv \lambda_{p5} p_{\tilde{p}\tilde{5}}$ contained in $Q^{[3]}$. For this purpose, we assign the following degrees to the members of the quartet:

$$\begin{aligned} \deg(p_{\tilde{p}\tilde{5}}) &= -2\,, \qquad & \deg(\theta_{p5}) &= +2\,, \\ \deg(\omega_{\tilde{p}\tilde{5}}) &= -1\,, \qquad & \deg(\lambda_{p5}) &= +1\,, \\ \deg(\text{rest}) &= 0\,. \end{aligned} \tag{4.37}$$

Under this grading, $\delta^{[3]}$ has degree -1 and $Q^{[3]}$ splits into

$$Q^{[3]} = \delta^{[3]} + Q_0^{[3]} + d_1^{[3]} + d_2^{[3]} + d_3^{[3]} + d_4^{[3]}\,, \tag{4.38}$$

where $(Q_0^{[3]}, d_n^{[3]})$ carry degrees $(0, n)$ respectively. What we wish to do is to retain the first two terms

$$\delta^{[3]} = \lambda_{p5} p_{\tilde{p}\tilde{5}}\,, \tag{4.39}$$

$$\begin{aligned} Q_0^{[3]} &= \left[\lambda^\alpha p_\alpha - \lambda_{p5} p_{\tilde{p}\tilde{5}}\right] - 4\left(c_{\tilde{5}} \partial x_5^+\right)_{\text{nzm}} - \lambda_{pq}\left(\theta_+ \theta_{\tilde{p}} \partial \theta_{\tilde{q}} + 2i\lambda_+^{-1} \theta_+ \lambda_{\tilde{p}} \partial x_{\tilde{q}}^-\right) - \\ &\quad - \theta_{pq}\left(\theta_+ \lambda_{\tilde{p}} \partial \theta_{\tilde{q}} - \theta_+ \theta_{\tilde{p}} \partial \lambda_{\tilde{q}} + \lambda_+ \theta_{\tilde{p}} \partial \theta_{\tilde{q}} - 2i\lambda_{\tilde{p}} \partial x_{\tilde{q}}^-\right) + c_0 T_0^{\text{EPS}}\,, \end{aligned} \tag{4.40}$$

and remove the rest carrying degree ≥ 1. Although it is straightforward to write down the explicit forms of $d_n^{[3]}$, what will be important is only their basic structure. Making explicit the dependence on λ_{p5} and θ_{p5}, we have

$$d_1^{[3]} = \lambda_{p5} F_{\tilde{p}}\,, \tag{4.41}$$

$$d_2^{[3]} = \theta_{p5} B_{\tilde{p}} + \lambda_{p5} \lambda_{q5} F'_{\tilde{p}\tilde{q}}\,, \tag{4.42}$$

$$d_3^{[3]} = \lambda_{p5} \theta_{q5} B_{\tilde{p}\tilde{q}} + \lambda_{p5} \partial \theta_{q5} \tilde{B}_{\tilde{p}\tilde{q}}\,, \tag{4.43}$$

$$d_4^{[3]} = \theta_{p5} \theta_{q5} F_{\tilde{p}\tilde{q}} + \theta_{p5} \partial \theta_{q5} \tilde{F}_{\tilde{p}\tilde{q}}\,, \tag{4.44}$$

where F's and B's are, respectively, fermionic and bosonic expressions free of the quartet members. Due to this property, they (anti-)commute with $\delta^{[3]}$ and with themselves. This fact will be utilized extensively below.

To remove $d_1^{[3]} \sim d_4^{[3]}$, we need to find an operator R such that

$$\begin{aligned} Q^{[3]} &= \delta^{[3]} + Q_0^{[3]} + d_1^{[3]} + d_2^{[3]} + d_3^{[3]} + d_4^{[3]} \\ &= e^{-R}\left(\delta^{[3]} + Q_0^{[3]}\right) e^R \end{aligned} \tag{4.45}$$

is realized. It is easy to see that R must start with degree two and hence its degree-wise decomposition can be written as $R = R_2 + R_3 + \cdots$. Then, the decomposition of the equation (4.45) with respect to the degree is given by

$$d_1^{[3]} = \left[\delta^{[3]}, R_2\right] , \tag{4.46}$$

$$d_2^{[3]} = \left[\delta^{[3]}, R_3\right] + \left[Q_0^{[3]}, R_2\right] , \tag{4.47}$$

$$d_3^{[3]} = \left[\delta^{[3]}, R_4\right] + \left[Q_0^{[3]}, R_3\right] + \frac{1}{2}\left[\left[\delta^{[3]}, R_2\right], R_2\right] , \tag{4.48}$$

$$d_4^{[3]} = \left[\delta^{[3]}, R_5\right]+\left[Q_0^{[3]}, R_4\right]+\frac{1}{2}\left[\left[\delta^{[3]}, R_2\right], R_3\right]+\frac{1}{2}\left[\left[\delta^{[3]}, R_3\right] R_2\right]+\frac{1}{2}\left[\left[Q_0^{[3]}, R_2\right], R_2\right] , \tag{4.49}$$

$$0 = \left[\delta^{[3]}, R_6\right] + \cdots , \tag{4.50}$$

and so on. Just as we did for the bosonic string, together with the relations that follow from the nilpotency of $Q^{[3]}$ we can solve these equations successively and determine R.

Let us illustrate the first step. To solve the equation (4.46) at degree 1, we look at the degree 0 part of $\left\{Q^{[3]}, Q^{[3]}\right\} = 0$, which reads $\{Q_0^{[3]}, Q_0^{[3]}\} + 2\{\delta^{[3]}, d_1^{[3]}\} = 0$. Since $Q_0^{[3]}$ itself is nilpotent, we must have $\{\delta^{[3]}, d_1^{[3]}\} = 0$. This suggests that $d_1^{[3]}$ may be written as $d_1^{[3]} = [\delta^{[3]}, R_2]$, with a suitable R_2. From the structure of $\delta^{[3]}$ and $d_1^{[3]}$ it is easy to see that $R_2 = \theta_{p5}F_{\tilde{p}}$ is the desired operator.

The procedure to solve the equations at higher degrees is similar: The nilpotency of $Q^{[3]}$ at degree $n-1$ suggests the existence of a solution to the equation (4.45) at degree n, and R_n can then be obtained using the specific structure of $d_n^{[3]}$'s given in (4.41)$\sim$ (4.44). Although this process is not in general guaranteed to terminate at finite steps, it does so in the present case, with the result

$$R = R_2 + R_3 + R_4 , \tag{4.51}$$

$$R_2 = \theta_{p5}F_{\tilde{p}} , \tag{4.52}$$

$$R_3 = \lambda_{p5}\theta_{q5}F'_{\tilde{p}\tilde{q}} , \tag{4.53}$$

$$R_4 = \frac{1}{2}\theta_{p5}\theta_{q5}B^A_{\tilde{p}\tilde{q}} + \frac{1}{2}\theta_{p5}\partial\theta_{q5}\tilde{B}^S_{\tilde{p}\tilde{q}} , \tag{4.54}$$

where the superscripts A and S signify the antisymmetric and the symmetric parts. Some details leading to this result are given in appendix B.2.

Having decoupled λ_{p5} and θ_{p5}, the next step is to decouple $(\lambda_{pq}, \omega_{\tilde{p}\tilde{q}}, \theta_{pq}, p_{\tilde{p}\tilde{q}})$. This set of fields form a quartet with respect to the operator $\delta^{[4]} \equiv \frac{1}{2}\lambda_{pq}p_{\tilde{p}\tilde{q}}$, contained in $Q^{[4]} \equiv \delta^{[3]} + Q_0^{[3]}$, which is the result of the previous step. The strategy should now be familiar. By assigning the degrees

$$\begin{aligned} \deg(\lambda_{pq}) &= 1 , & \deg(\omega_{\tilde{p}\tilde{q}}) &= -1 , \\ \deg(\theta_{pq}) &= 2 , & \deg(p_{\tilde{p}\tilde{q}}) &= -2 , \\ \deg(\text{rest}) &= 0 , \end{aligned} \tag{4.55}$$

we can decompose $Q^{[4]}$ into

$$Q^{[4]} = \delta^{[4]} + Q_0^{[4]} + d_1^{[4]} + d_2^{[4]} , \tag{4.56}$$

where $\deg(\delta^{[4]}, Q^{[4]}, d_n^{[4]}) = (-1, 0, n)$. The explicit forms of these structures are

$$\delta^{[4]} = \frac{1}{2}\lambda_{pq} p_{\tilde{p}\tilde{q}} \,, \tag{4.57}$$

$$Q_0^{[4]} = \left[\lambda^\alpha p_\alpha - \frac{1}{2}\lambda_{pq} p_{\tilde{p}\tilde{q}}\right] - 4(c_{\tilde{5}}\partial x_{\tilde{5}}^-)_{\rm nzm} + c_0 T_0^{\rm EPS} \,, \tag{4.58}$$

$$d_1^{[4]} = -\lambda_{pq}\left(\theta_+\theta_{\tilde{p}}\partial\theta_{\tilde{q}} + 2i\lambda_+^{-1}\theta_+\lambda_{\tilde{p}}\partial x_{\tilde{q}}^-\right) \,, \tag{4.59}$$

$$d_2^{[4]} = -\theta_{pq}\left(\theta_+\lambda_{\tilde{p}}\partial\theta_{\tilde{q}} - \theta_+\theta_{\tilde{p}}\partial\lambda_{\tilde{q}} + \lambda_+\theta_{\tilde{p}}\partial\theta_{\tilde{q}} - 2i\lambda_{\tilde{p}}\partial x_{\tilde{q}}^-\right) . \tag{4.60}$$

At this stage, it is gratifying to note that the sum $\delta^{[4]} + Q_0^{[4]}$ is precisely the expression $\tilde{Q}$ that we want. Thus, the only remaining task is to show that $d_1^{[4]}$ and $d_2^{[4]}$ can be removed by a similarity transformation of the form

$$Q^{[4]} = \delta^{[4]} + Q_0^{[4]} + d_1^{[4]} + d_2^{[4]} = e^{-S}\left(\delta^{[4]} + Q_0^{[4]}\right)e^S = e^{-S}\tilde{Q}e^S \,. \tag{4.61}$$

by a suitable operator S. A straightforward computation shows that $d_1^{[4]}$ and $d_2^{[4]}$ can be written as

$$d_1^{[4]} = \left[\delta^{[4]}, S\right] , \qquad d_2^{[4]} = \left[Q_0^{[4]}, S\right] , \tag{4.62}$$

where S, carrying degree 2 under the current grading, is given by

$$S = \frac{1}{2}\theta_{pq} F_{\tilde{p}\tilde{q}} \,, \tag{4.63}$$

$$F_{\tilde{p}\tilde{q}} = -\left(\theta_{\tilde{p}}\partial\theta_{\tilde{q}} - \theta_{\tilde{q}}\partial\theta_{\tilde{p}}\right)\theta_+ + 2i\lambda_+^{-1}\theta_+\left(\lambda_{\tilde{p}}\partial x_{\tilde{q}}^- - \lambda_{\tilde{q}}\partial x_{\tilde{p}}^-\right) . \tag{4.64}$$

Furthermore, it is easy to confirm that the double commutators all vanish:

$$\left[\left[\delta^{[4]}, S\right], S\right] = \left[\left[Q_0^{[4]}, S\right], S\right] = 0 \,. \tag{4.65}$$

Combining these results, the similarity transformation (4.61) is proved.

To summarize, we have demonstrated that the BRST operator $\hat{Q}$ for the EPS formalism can be mapped to $\bar{Q}$, the one for the light-cone GS formalism, by a series of similarity transformations in the manner

$$\bar{Q} = e^{c_0 K} e^T e^S e^R e^Z e^Y e^X \hat{Q} e^{-X} e^{-Y} e^{-Z} e^{-R} e^{-S} e^{-T} e^{-c_0 K} \,. \tag{4.66}$$

4.3 Zero modes and cohomology

Having reduced $\hat{Q}$ to $\bar{Q} = \delta_{lc} + Q_{lc}$, we now discuss in some detail the cohomology of the latter operator, focusing in particular on the zero mode sector.

Let us write the general state of our system as $|\Phi\rangle \otimes |\Psi\rangle$, where $|\Phi\rangle$ consists of the light-cone fields $p^\pm, \alpha_n^i, S_n^a$ and the ghost zero mode c_0, while $|\Psi\rangle$ is composed of the remaining fields. First, states with non-zero mode excitations in $|\Psi\rangle$ are unphysical by the standard argument: As discussed previously, $\left\{\bar{Q}, K\right\} = \{\delta_{lc}, K\} = N$ holds with N being the level-counting operator for the Ψ-modes, and non-trivial cohomology of $\bar{Q}$ occurs only in the sector where $N = 0$ i.e. without non-zero Ψ modes. We shall denote such sector as $|\Phi\rangle \otimes |\Psi_0\rangle$, where the subscript 0 signifies that only the zero modes are present.

Now in this sector, $\bar{Q}$ is effectively reduced to $\bar{Q}_0 \equiv \delta_{lc,0} + Q_{lc}$, where $\delta_{lc,0}$ is the zero mode part of δ_{lc} given by

$$\delta_{lc,0} = \lambda_0^{\dot{a}} p_{\dot{a},0} - 2\lambda_0^a \tilde{S}_0^{-a} \,. \tag{4.67}$$

This operator serves as the nilpotent operator with respect to which $(\lambda_0^{\dot{a}}, \omega_{\dot{a},0}, \theta_0^{\dot{a}}, p_{\dot{a},0})$ and $(\lambda_0^a, \omega_{a,0}, \tilde{S}_0^{+a}, \tilde{S}_0^{-a})$ form quartets. To make the discussion transparent, let us denote each of these zero-mode quartets generically as (γ, β, c, b), the members of which satisfy the (anti-)commutation relations[5]

$$[\gamma, \beta] = 1 \,, \qquad \{c, b\} = 1 \,. \tag{4.68}$$

In the γ-c representation of the states, the role of $\delta_{lc,0}$ is played by the operator

$$q \equiv \gamma b = \gamma \frac{\partial}{\partial c} \,, \tag{4.69}$$

and we will first study the cohomology of this nilpotent "BRST" operator. This depends on the choice of the Hilbert space. Most notably, if we allow a special operator of the form $\xi \equiv \gamma^{-1} c$, then since $\{q, \xi\} = 1$ the cohomology becomes trivial. Indeed any q-closed state ψ can be written as $\psi = \{q, \xi\}\, \psi = q\xi\psi + \xi q \psi = q(\xi\psi)$, which is q-exact. Thus we need to exclude ξ from our Hilbert space. Actually, in the case of $q = \lambda_0^{\dot{a}} p_{\dot{a},0}$, exclusion is automatic if we demand SO(8) covariance. However, as our similarity transformation only respects U(4) covariance in the intermediate steps, it is safe and do no harm to exclude such an operator explicitly. This however does not mean that other operators with γ^{-1} factor need be excluded, since they do not trivialize the cohomology as ξ does. In fact we need such operators, for instance $Y = -2i\lambda_+^{-1} c_{\tilde{p}} d_p$ in (4.30), in the similarity transformation. Now for γ^{-1} factor to be admissible, we need to exclude states which are localized at $\gamma = 0$ namely ones with $\delta(\gamma)$ factor, on which γ^{-1} is ill-defined.

With the above specification of the Hilbert space, let us analyze the cohomology of q. The most general allowed wave function is of the form

$$\psi = f(\gamma) + \tilde{f}(\gamma) c \,, \tag{4.70}$$

where $f(\gamma)$ and $\tilde{f}(\gamma)$ are arbitrary functions not localized at $\gamma = 0$ and in addition $\tilde{f}(\gamma)$ should not be proportional to γ^{-1} to avoid the structure $\gamma^{-1} c$. Then q-closedness condition reads $q\psi = \gamma \tilde{f}(\gamma) = 0$. Since we exclude the solution $\tilde{f} \sim \delta(\gamma)$, this demands $\tilde{f}(\gamma) = 0$ and $\psi = f(\gamma)$. On the other hand the most general q-exact state is of the form $q\chi$ with $\chi = g(\gamma) + \tilde{g}(\gamma) c$, where the same restrictions as for $f, \tilde{f}$ apply to g and $\tilde{g}$. Since $q\chi = \gamma \tilde{g}(\gamma)$, q-closed ψ is q-exact if $f(\gamma)$ can be written as $\gamma \tilde{g}(\gamma)$. Since $\tilde{g}(\gamma) \neq \gamma^{-1}$ this means that all q-closed ψ, except $\psi = 1$, are trivial. Hence we find that the cohomology of q in our Hilbert space consists solely of the vacuum state annihilated by β and b.

Now we apply this result to our original problem and study the cohomology of $\bar{Q}$. $\bar{Q}$-closed state of the form $|\Phi\rangle \otimes |\Psi_0\rangle$ must satisfy $Q_{lc}|\Phi\rangle = 0$ and $\delta_{lc}|\Psi_0\rangle = 0$. If $|\Psi_0\rangle$ is not the vacuum state $|0\rangle$, it can be written as $\delta_{lc,0}|\Xi_0\rangle$. But then, $|\Phi\rangle \otimes |\Psi_0\rangle = (Q_{lc} + \delta_{lc,0})|\Phi\rangle \otimes |\Xi_0\rangle$

[5] For instance, (b, c) ghosts here stand for $(\theta^{\dot{a}}, p_{\dot{a},0})$ or $(\tilde{S}_0^{+a}, \tilde{S}_0^{-a})$. Although the symbol (b, c) have already been used to denote $(b_5, c_{\tilde{5}})$ pair, we believe there will be no confusion.

and such a state is cohomologically trivial. Hence the cohomology of $\bar{Q}$ is reduced to that of Q_{lc} in the space $|\Phi\rangle \otimes |0\rangle$. The rest of the analysis is entirely parallel to the bosonic case and we reproduce the light-cone on-shell spectrum.

5. Discussions

In this article, making use of the systematic technique developed in our previous work [30], we have been able to construct an explicit quantum mapping between the light-cone quantized GS string and the extended version of the pure spinor formalism in the form of a similarity transformation. As already remarked in the introduction, this allows one to see the mechanism of the decoupling of unphysical degrees of freedom in a transparent way, which is an improvement over the earlier work [15] requiring infinite number of ghosts.

There are of course many further works to be performed. An immediate question of interest is to study how some basic operators of the LCGS are represented in the EPS and vice versa. For instance, it would be intriguing to see the counterparts of the physical oscillators α_n^i and S_n^a on the EPS side, which are the analogues of the DDF operators. They are expected to be closely related to the massless vertex operators already constructed in [29]. Another obvious project is to understand how the supercharges of the two formulations are mapped into each other. This may not be quite straightforward since the supercharges in the LCGS act only on the restricted physical space satisfying the on-shell condition $L_0^{lc}|\phi\rangle = 0$, whereas the ones in the EPS act on all the components of x^μ and θ^α including the unphysical ones. In the BRST framework, this means that we must compare the action of the supercharges and their closure relations at the cohomological level, i.e. up to BRST-exact terms. Once these correspondences are clarified, it will become possible to map the amplitude calculation as well. This would clarify the rules of computations on the EPS side, which have not yet been spelled out.

More challenging task is the extraction of information on the underlying local symmetry of our extended theory, crucial for the eventual construction of the fundamental action. As we have identified the structure of all the quartets in this work, this knowledge should give some useful hints. In this regard, let us recall that the reparametrization (b, c) ghosts appeared as part of the ghosts $(b_P, c_{\tilde{P}})$ which are introduced to remove the PS constraints. This suggests that in this type of formulation bosonic and fermionic local symmetries are intimately intertwined. Clarification of this structure certainly deserves further investigation.

The problems listed above and more are currently under study and we hope to report our progress elsewhere.

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A. Conventions and useful formulas

In this appendix, our conventions and some useful formulas are collected.

A.1 Spinors and Γ-matrices in real basis

32×32 SO(9,1) Gamma matrices are denoted by Γ^μ, ($\mu = 0,1,\ldots,9$) and obey the Clifford algebra $\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}$. Our metric convention is $\eta^{\mu\nu} = (-,+,+,\ldots,+)$. The 10-dimensional chirality operator is taken to be $\bar\Gamma_{10} = -\Gamma^0\Gamma^1\cdots\Gamma^9$ and it satisfies $\bar\Gamma_{10}^2 = 1$.

In the Majorana or real basis (R-basis for short), Γ^μ are all real and unitary. Within the R-basis, we define the Weyl basis to be the one in which $\bar\Gamma_{10} = \mathrm{diag}(1_{16}, -1_{16})$, where 1_{16} is the 16×16 unit matrix. In this basis, a general 32-component spinor Λ is written as $\Lambda = \begin{pmatrix}\lambda^\alpha\\ \lambda_\alpha\end{pmatrix}$, where λ^α and λ_α are chiral and anti-chiral respectively, with $\alpha = 1 \sim 16$. Correspondingly, Γ^μ, which flips chirality, takes the structure

$$\Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)^{\alpha\beta} \\ (\gamma^\mu)_{\alpha\beta} & 0 \end{pmatrix}, \tag{A.1}$$

where the 16×16 γ-matrices $(\gamma^\mu)^{\alpha\beta}$ and $(\gamma^\mu)_{\alpha\beta}$ are real symmetric and satisfy

$$(\gamma^\mu)_{\alpha\beta}(\gamma^\nu)^{\beta\gamma} + (\gamma^\nu)_{\alpha\beta}(\gamma^\mu)^{\beta\gamma} = 2\eta^{\mu\nu}\delta_\alpha^\gamma\,. \tag{A.2}$$

For $\gamma^m, m = 1\sim 9$, we have $(\gamma^m)^{\alpha\beta} = (\gamma^m)_{\alpha\beta}$. As for γ^0, we will use the convention $(\gamma^0)_{\alpha\beta} = -(\gamma^0)^{\alpha\beta} = \delta_{\alpha\beta}$.

A.2 U(5) basis

The spinor representations for SO(9,1) and SO(10) can be conveniently constructed with the use of 5 pairs of fermionic oscillators $(b_P, b_P^\dagger)_{P=1\sim 5}$ satisfying the anti-commutation relations $\left\{b_P, b_Q^\dagger\right\} = \delta_{PQ}$. States are built upon the oscillator vacuum, to be denoted by $|+\rangle$, annihilated by all the b_P's. b_P and $b_P^\dagger$ transform respectively as 5 and $\bar 5$ of U(5) subgroup[6]. Γ^μ matrices can then be regarded as linear operators in this Fock space and in the case of SO(9,1) they are identified as

$$\Gamma^{2p} = \frac{1}{i}(b_p - b_p^\dagger)\,, \quad \Gamma^{2P-1} = b_P + b_P^\dagger\,, \tag{A.3}$$

$$\Gamma^0 = -(b_5 - b_5^\dagger)\,, \tag{A.4}$$

Here and throughout, we use the notation $P = (p,5)$, where the lower case letter p runs from 1 to 4.

The states built upon $|+\rangle$ and their conjugates are defined as

$$|P_1P_2\ldots P_k\rangle \equiv b_{P_1}^\dagger\cdots b_{P_k}^\dagger|+\rangle\,, \qquad \langle P_1P_2\ldots P_k| \equiv \langle +|b_{P_k}b_{P_{k-1}}\cdots b_{P_1}\,. \tag{A.5}$$

[6]For SO(9,1), it is a Wick rotation of U(5) but we continue to use this terminology following common usage.

Further, we define

$$|-\rangle \equiv b_1^\dagger b_2^\dagger \ldots b_5^\dagger|+\rangle = \frac{1}{5!}\epsilon_{P_1P_2\ldots P_5} b_{P_1}^\dagger b_{P_2}^\dagger \cdots b_{P_5}^\dagger|+\rangle\,, \tag{A.6}$$

$$|\tilde{P}_1\ldots\tilde{P}_k\rangle \equiv \frac{1}{(5-k)!}\epsilon_{P_1\ldots P_kQ_{k+1}\ldots Q_5}|Q_{k+1}\ldots Q_5\rangle\,, \tag{A.7}$$

and their corresponding conjugates, where $\epsilon_{12345}\equiv 1$. These states satisfy the orthonormality relations

$$\langle +|+\rangle = \langle -|-\rangle = 1\,, \tag{A.8}$$

$$\langle P_1\ldots P_k|Q_1\ldots Q_k\rangle = \langle \tilde{P}_1\ldots\tilde{P}_k|\tilde{Q}_1\ldots\tilde{Q}_k\rangle = \delta^{P_1\ldots P_k}_{Q_1\ldots Q_k}\,. \tag{A.9}$$

In this basis, chiral and anti-chiral spinors can be written as

$$\text{chiral:}\qquad |\lambda\rangle = \lambda_+|+\rangle + \frac{1}{2}\lambda_{PQ}|PQ\rangle + \lambda_{\tilde{P}}|\tilde{P}\rangle\,, \tag{A.10}$$

$$\text{anti-chiral:}\qquad |\psi\rangle = \psi_-|-\rangle + \frac{1}{2}\psi_{\tilde{P}\tilde{Q}}|\tilde{P}\tilde{Q}\rangle + \psi_P|P\rangle\,. \tag{A.11}$$

The charge conjugation matrix in this basis is given by

$$C = -(b_5-b_5^\dagger)(b_1-b_1^\dagger)\cdots(b_4-b_4^\dagger) = \Gamma^0\Gamma^2\Gamma^4\Gamma^6\Gamma^8\,, \tag{A.12}$$

and satisfies $C^2=-1$. Its action on the states is

$$C|+\rangle = |-\rangle\,,\quad C|PQ\rangle = -|\tilde{P}\tilde{Q}\rangle\,,\quad C|\tilde{P}\rangle = |P\rangle\,, \tag{A.13}$$

$$C|-\rangle = -|+\rangle\,,\quad C|\tilde{P}\tilde{Q}\rangle = |PQ\rangle\,,\quad C|P\rangle = -|\tilde{P}\rangle\,. \tag{A.14}$$

A.3 SO(8) parametrization

In the following, λ^α and χ_α denote 10D chiral and anti-chiral spinors respectively. 16×16 SO(8) chiral projectors are defined by

$$P_8^\pm = \frac{1}{2}(1\pm\gamma^9)\,,\qquad \gamma^9 = \gamma^1\gamma^2\cdots\gamma^8\,,\qquad (\gamma^9)^2 = 1\,. \tag{A.15}$$

In the R-basis, we may take γ^9 to be diagonal i.e. $\gamma^9 = \text{diag}\,(1_8,-1_8)$. Then, the decomposition of χ^α and ψ_α is simply given by

$$\lambda^\alpha = \begin{pmatrix}\lambda^a\\ \lambda^{\dot a}\end{pmatrix},\qquad \chi_\alpha = \begin{pmatrix}\chi_a\\ \chi_{\dot a}\end{pmatrix}. \tag{A.16}$$

In this representation, γ^i ($i=1\sim 8$) can be written as

$$\gamma^i = \begin{pmatrix}0 & \gamma^i_{a\dot a}\\ \gamma^i_{\dot a a} & 0\end{pmatrix}, \tag{A.17}$$

where $\gamma^i_{a\dot a}$ are symmetric SO(8) Pauli matrices satisfying the anti-commutation relation and the Fierz identity:

$$\gamma^i_{a\dot a}\gamma^j_{\dot a b} + \gamma^j_{a\dot a}\gamma^i_{\dot a b} = 2\delta^{ij}\delta_{ab}\,, \tag{A.18}$$

$$\gamma^i_{a\dot a}\gamma^i_{b\dot b} + \gamma^i_{a\dot b}\gamma^i_{\dot a b} = 2\delta_{ab}\delta_{\dot a\dot b}\,. \tag{A.19}$$

$P_8^\pm$ are essentially $\gamma^\pm$, which are the off-diagonal elements of $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)$. Precise relations are

$$(P_8^\pm)_{\alpha\beta} = \frac{1}{\sqrt{2}}\gamma^\pm_{\alpha\beta}\,, \qquad (P_8^\mp)^{\alpha\beta} = -\frac{1}{\sqrt{2}}\gamma^{\pm\alpha\beta}\,. \tag{A.20}$$

In the R-basis, $(P_8^+)_{ab} = \delta_{ab}$ and $(P_8^-)_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}}$, so that raising and lowering of chiral and anti-chiral indices a and $\dot{a}$ are trivial. However, in the U(5) basis, we must use $\pm\gamma^\pm/\sqrt{2}$ for this purpose, the action of which is non-trivial.

For the construction of the similarity transformation for the superstring case described in section 4, we often need the expression of SO(8) spinors in terms of U(5) components. It is convenient to introduce the following abbreviated ket notations:

$$|\lambda\rangle \equiv |\lambda^a\rangle\,, \qquad |\bar{\lambda}\rangle \equiv |\lambda^{\dot{a}}\rangle\,, \tag{A.21}$$

$$|\chi\rangle \equiv |\chi_a\rangle\,, \qquad |\bar{\chi}\rangle \equiv |\chi_{\dot{a}}\rangle\,. \tag{A.22}$$

Then, their U(4) decompositions are given by

$$|\lambda\rangle = \lambda_+|+\rangle + \frac{1}{2}\lambda_{pq}|pq\rangle + \lambda_{\tilde{5}}|\tilde{5}\rangle\,, \tag{A.23}$$

$$|\bar{\lambda}\rangle = \lambda_{p5}|p5\rangle + \lambda_{\tilde{p}}|\tilde{p}\rangle\,, \tag{A.24}$$

$$|\chi\rangle = \chi_-|-\rangle + \frac{1}{2}\chi_{\tilde{p}\tilde{q}}|\tilde{p}\tilde{q}\rangle + \chi_5|5\rangle\,, \tag{A.25}$$

$$|\bar{\chi}\rangle = \chi_q|q\rangle + \chi_{\tilde{q}\tilde{5}}|\tilde{q}\tilde{5}\rangle\,. \tag{A.26}$$

Together with the expression of Γ-matrices in terms of the fermionic oscillators $b_P, b_P^\dagger$ given in (A.3) and (A.4), one can convert various spinor bilinears from SO(8) parametrization to U(5) parametrizations. We just give one such example for illustration. Let λ^a and θ^a be 10D chiral SO(8) chiral spinors and consider the bilinear $\lambda^a\theta_a$ in the R-basis. As already remarked, in going to other basis we must interpret this as $\lambda^a(P_8^+)_{ab}\theta^b = \lambda\gamma^+\theta/\sqrt{2}$. To compute this in U(5) basis, we should write this as $\langle\lambda|C\Gamma^+|\theta\rangle/\sqrt{2}$ and use the identification $\Gamma^+ = \sqrt{2}\,b_5^\dagger$. Then, using the known action of $b_5^\dagger$ and C on U(5) states, we easily obtain the result $\lambda^a\theta_a = -\lambda_+\theta_{\tilde{5}} - \lambda_{\tilde{5}}\theta_+ + (1/4)\epsilon_{pqrs5}\lambda_{rs}\theta_{pq}$.

B. Some details of the construction of similarity transformations

In this appendix, we supply some further details of the construction of the similarity transformations.

B.1 Bosonic string

Below we shall complete the proof of the similarity transformation (3.39), reorganized according to the degree as

$$\begin{aligned} Q &= \delta + Q_0 + d_1 + d_2 + d_3 = e^{-R}(\delta + Q_0)e^R \\ &= \delta + Q_0 + [\delta, R] + [Q_0, R] + \frac{1}{2}\,[[\delta, R,]\,R] + \frac{1}{2}\,[[Q_0, R,]\,R] + \cdots \\ &= \delta + Q_0 + [\delta, R_2] + ([\delta, R_3] + [Q_0, R_2]) + \left([Q_0, R_3] + \frac{1}{2}\,[[\delta, R_2,]\,R_2]\right) + \cdots\,. \end{aligned} \tag{B.1}$$

R is given, as in (3.40), by

$$R = R_2 + R_3\,, \qquad R_2 = \left\{\hat{K}, d_1\right\}\,, \qquad R_3 = \frac{1}{2}\left\{\hat{K}, d_2\right\}\,, \tag{B.2}$$

which has been determined to satisfy $[\delta, R_2 + R_3] = d_1 + d_2$. Throughout, our strategy is to make use of the relations (3.22) $\sim$ (3.30) which follow from the nilpotency of Q as well as the representations of R given in (B.2) and appropriate graded Jacobi identities to circumvent explicit computations involving the rather complicated operator d_1.

To verify (B.1) up to degree 2, we need to show that $[Q_0, R_2]$ vanishes. Using (B.2) it can be computed as

$$[Q_0, R_2] = \left[Q_0, \left\{\hat{K}, d_1\right\}\right] = \left[\left\{Q_0, \hat{K}\right\}, d_1\right] - \left[\hat{K}, \{Q_0, d_1\}\right]\,. \tag{B.3}$$

The last term on the RHS vanishes by the first equation in (3.37), while it is easy to check explicitly that $\left\{Q_0, \hat{K}\right\} = 0$. Thus we get $[Q_0, R_2] = 0$.

Next we move on to degree 3. First, by using $R_2 = \left\{\hat{K}, d_1\right\}$ and graded Jacobi identities, the double commutator $[[\delta, R_2,] R_2]$ can easily be rewritten as $[[\delta, R_2,] R_2] = -\frac{1}{2}\left[\hat{K}, \{d_1, d_1\}\right]$. Applying the relation (E_2) in (3.26), we may write this as

$$[[\delta, R_2], R_2] = \left[\hat{K}, \{\delta, d_3\}\right] + \left[\hat{K}, \{Q_0, d_2\}\right]\,. \tag{B.4}$$

The first term can be directly computed to give $2d_3$, while the second term, by using a Jacobi identity and the relation $\left\{Q_0, \hat{K}\right\} = 0$, can be shown to equal $-2\,[Q_0, R_3]$. Combining these results, we get

$$\frac{1}{2}\,[[\delta, R_2], R_2] + [Q_0, R_3] = d_3\,, \tag{B.5}$$

which proves the validity of the similarity transformation at degree 3. The remaining task is to show that terms with higher degrees all vanish.

For this purpose it is convenient to derive some relations at degree 3 and 4. First from the structure of Q_0 and d_3 it is easy to see that $\{Q_0, d_3\} = 0$. Due to the relation (E_3) shown in (3.27), this in turn dictates $\{d_2, d_1\} = 0$. Similarly, $\{d_2, d_2\} = 0$ by inspection and this leads via the relation (E_4) in (3.28) to $\{d_3, d_1\} = 0$. This then leads to $[R_3, d_1] = 0$ since, apart from a b_0 factor, R_3 is actually proportional to d_3. Other useful vanishing relations at these degrees are: $\left\{\hat{K}, R_2\right\} = 0$, which follows from the nilpotency of $\hat{K}$, and $\left\{\hat{K}, d_3\right\} = \left\{\hat{K}, R_3\right\} = 0$, which can be checked explicitly.

Now we are ready to show that the double commutators $[[\delta, R_i], R_j]$ and $[[Q_0, R_i], R_j]$ with $i = 2, 3$, except $[[\delta, R_2], R_2]$, and the triple commutators $[[[\delta, R_2], R_2], R_i]$ vanish. First, using $\{Q_0, d_3\} = 0$, one easily finds that $[Q_0, R_3] \propto L_0^{\rm tot} R_3$ and hence $[[Q_0, R_3], R_3] = 0$. Since we already know $[Q_0, R_2]$ vanishes, this shows that $[[Q_0, R_i], R_j]$ all vanish. Next we can prove a somewhat non-trivial relation $[d_2, R_2] = 0$ by rewriting $[d_2, R_2]$ via a Jacobi identity as $2\,[R_3, d_1] - \left[\hat{K}, \{d_2, d_1\}\right]$. Since $d_2 = [\delta, R_3]$, this is nothing but the vanishing of the double commutator $[[\delta, R_3], R_2] = 0$. This in turn implies $[[\delta, R_2], R_3] = 0$ with the aid of a Jacobi identity, since $[R_3, R_2] = -\frac{1}{2}\left\{\left[\hat{K}, R_2\right], d_2\right\} - \frac{1}{2}\left\{\hat{K}, [d_2, R_2]\right\} = 0$.

Also, $[[\delta, R_3], R_3] = [d_2, R_3] = 0$ is seen to hold by inspection. Finally, by using (B.5) and $[d_3, R_2] = 0$ which follows from previously derived vanishing relations, we find that the triple commutators $[[[\delta, R_2], R_2], R_i]$ vanish. Thus, the similarity transformation terminates at degree 3 and our assertion is proved.

B.2 Superstring

In this appendix, we shall supply some details of the construction of the operator R sketched in subsection 4.2. To facilitate the discussion, we shall denote the graded commutator $[A, B\}$ of two integrated operators by AB and suppress the superscript "[3]" on $\delta^{[3]}$, $Q_0^{[3]}$ and $d_n^{[3]}$. In this notation, the equations (4.46) $\sim$ (4.50) take the form

$$(\tilde{E}_1): \quad d_1 = \delta R_2 \,, \tag{B.6}$$

$$(\tilde{E}_2): \quad d_2 = \delta R_3 + Q_0 R_2 \,, \tag{B.7}$$

$$(\tilde{E}_3): \quad d_3 = \delta R_4 + Q_0 R_3 + \frac{1}{2}(\delta R_2) R_2 \,, \tag{B.8}$$

$$(\tilde{E}_4): \quad d_4 = \delta R_5 + Q_0 R_4 + \frac{1}{2}(\delta R_2) R_3 + \frac{1}{2}(\delta R_3) R_2 + \frac{1}{2}(Q_0 R_2) R_2 \,, \tag{B.9}$$

$$(\tilde{E}_5): \quad 0 = \delta R_6 + Q_0 R_5 + \cdots \,. \tag{B.10}$$

Also it will be useful to display the explicit equations which follow from the nilpotency of $Q^{[3]}$. They read

$$(N_0): \quad 0 = 2\delta d_1 + (Q_0)^2, \tag{B.11}$$

$$(N_1): \quad 0 = \delta d_2 + Q_0 d_1, \tag{B.12}$$

$$(N_2): \quad 0 = \delta d_3 + Q_0 d_2, \tag{B.13}$$

$$(N_3): \quad 0 = \delta d_4 + Q_0 d_3, \tag{B.14}$$

$$(N_4): \quad 0 = Q_0 d_4 \,. \tag{B.15}$$

As already mentioned in the text, the fact that F's and B's (anti-)commute among themselves will be tacitly used throughout the subsequent analysis.

Since we have already described the solution of $(\tilde{E}_1)$ in the text, we begin with $(\tilde{E}_2)$, which can be written as $d_2 - Q_0 R_2 = \delta R_3$. To determine R_3, let us compute $d_2 - Q_0 R_2$ more concretely. Substituting the explicit forms of d_2, Q_0 and R_2, we obtain

$$d_2 - Q_0 R_2 = \theta_{p5}(B_{\tilde{p}} + Q_0 F_{\tilde{p}}) + \lambda_{p5}\lambda_{q5} F'_{\tilde{p}\tilde{q}} \,. \tag{B.16}$$

Actually one can show that the first term on the RHS vanishes due to the nilpotency relation (N_1): Substituting $d_1 = \lambda_{p5} F_{\tilde{p}}$ and $d_2 = \theta_{p5} B_{\tilde{p}} + \lambda_{p5}\lambda_{q5} F'_{\tilde{p}\tilde{q}}$, (N_1) reads

$$0 = \delta d_2 + Q_0 d_1 = \lambda_{p5}(B_{\tilde{p}} + Q_0 F_{\tilde{p}}) \,. \tag{B.17}$$

Since λ_{p5}'s are algebraically independent, $B_{\tilde{p}} + Q_0 F_{\tilde{p}}$ itself must vanish. In this way the equation $d_2 - Q_0 R_2 = \delta R_3$ above is reduced to $\lambda_{p5}\lambda_{q5} F'_{\tilde{p}\tilde{q}} = \delta R_3$. Now it can easily be solved for R_3 as

$$R_3 = \lambda_{p5}\theta_{q5} F'_{\tilde{p}\tilde{q}} \,. \tag{B.18}$$

Next, we move on to the degree 3 equation $(\tilde{E}_3)$. From the structure of R_2 given in equation (4.52), it is easy to see that the double commutator term $\frac{1}{2}(\delta R_2)R_2$ vanishes. Hence, $(\tilde{E}_3)$ simplifies to

$$d_3 - Q_0 R_3 = \delta R_4 \,. \tag{B.19}$$

where the explicit form of the LHS is worked out as

$$d_3 - Q_0 R_3 = \lambda_{p5}\theta_{q5}\left(B_{\tilde{p}\tilde{q}} + Q_0 F'_{\tilde{p}\tilde{q}}\right) + \lambda_{p5}\partial\theta_{q5}\tilde{B}_{\tilde{p}\tilde{q}} \,. \tag{B.20}$$

To see that this expression is actually δ-exact, we need the information from the nilpotency relation (N_2). Upon substituting $d_2 = Q_0 R_2 + \delta R_3$, (N_2) becomes

$$0 = \delta d_3 + Q_0 d_2 = \lambda_{p5}\lambda_{q5}B_{\tilde{p}\tilde{q}} + \lambda_{p5}\partial\lambda_{q5}\tilde{B}_{\tilde{p}\tilde{q}} + \theta_{p5}\theta_{q5}\left(Q_0 F'_{\tilde{p}\tilde{q}}\right) + \lambda_{p5}\lambda_{q5}\left(Q_0 F'_{\tilde{p}\tilde{q}}\right) \tag{B.21}$$

$$= \lambda_{p5}\lambda_{q5}\left(B^S_{\tilde{p}\tilde{q}} + Q_0 F'_{\tilde{p}\tilde{q}} - \frac{1}{2}\partial\tilde{B}^S_{\tilde{p}\tilde{q}}\right) + \frac{1}{2}(\lambda_{p5}\partial\lambda_{q5} - \lambda_{q5}\partial\lambda_{p5})\tilde{B}^A_{\tilde{p}\tilde{q}} \,, \tag{B.22}$$

where the superscripts S and A stand for the symmetric and antisymmetric parts, respectively. From this we immediately obtain $B^S_{\tilde{p}\tilde{q}} + Q_0 F'_{\tilde{p}\tilde{q}} = \frac{1}{2}\partial\tilde{B}^S_{\tilde{p}\tilde{q}}$ and $\tilde{B}^A_{\tilde{p}\tilde{q}} = 0$. Using these relations (B.20) is reduced to

$$d_3 - Q_0 R_3 = \lambda_{p5}\theta_{q5}\left(B^A_{\tilde{p}\tilde{q}} + \frac{1}{2}\partial\tilde{B}^S_{\tilde{p}\tilde{q}}\right) + \lambda_{p5}\partial\theta_{q5}\tilde{B}^S_{\tilde{p}\tilde{q}} \,. \tag{B.23}$$

It is now not difficult to see that the RHS can be written as δR_4, where

$$R_4 = \frac{1}{2}\theta_{p5}\theta_{q5}B^A_{\tilde{p}\tilde{q}} + \frac{1}{2}\theta_{p5}\partial\theta_{q5}\tilde{B}^S_{\tilde{p}\tilde{q}} \,. \tag{B.24}$$

In an entirely similar manner we can analyze $(\tilde{E}_n)$ to obtain R_{n+1} for $n \geq 4$. For $n = 4$, the double commutator terms vanish as in previous steps and $(\tilde{E}_4)$ becomes $d_4 - Q_0 R_4 = \delta R_5$. This time a simplifying feature sets in at this stage. Analysis of $d_4 - Q_0 R_4$ and the relation $0 = \delta R_4 + Q_0 R_3$ reveals that $d_4 - Q_0 R_4$ itself vanishes and hence we get $R_5 = 0$. Above $n = 5$, one can easily check that $R_{n\geq 6} = 0$ solve all the equations. This completes the construction of R.

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